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A greater man than any of us said

> The evil that men do lives after them
> The good is oft interréd with their bones.

Let us begin, therefore, with some criticisms. And then let us study not only those of Pearson's contributions to science and culture which are widely known, but perhaps some also which should be disinterred and brought once more into the light of day.

As Pitt first stated, and Acton restated more precisely, all power corrupts. It is impossible to be a professor in charge of an important department, and the editor of an important journal, without being somewhat corrupted. We can now see that in both capacities Pearson made mistakes. He rejected lines of research which later turned out to be fruitful. He used his own energy and that of his subordinates in research which turned out to be much less important than he believed. It is, however, very easy to say what any one ought to have done fifty years ago!

But this criticism can be, and has been, pushed much further. It is said that Pearson espoused a fundamentally false theory of heredity, and therefore of evolution, and that as a consequence his work was not merely useless, but actually retarded progress. Had Pearson become dictator of British research on heredity and evolution, this might have been true. Fortunately he did not. I believe that his theory of heredity was incorrect in some fundamental respects. So was Columbus' theory of geography. He set out for China, and discovered America. But he is not regarded as a failure for this reason. When I turn to Pearson's great series of papers on the mathematical theory of evolution, published in the last years of the nineteenth century, I find that the theories of evolution now most generally accepted are very far from his own. *But* I find that in the search for a self-consistent theory of evolution he devised methods which are not only indispensable in any discussion of evolution. They are essential in every serious application of statistics to any problem whatever. If, for example, I wish to describe the distribution of British incomes, the response of different individuals to a drug, or the results of testing materials used in engineering, I must start off from the foundation laid in his memoir on 'Skew variation in homogeneous material'. After sixty-three years I shall certainly take some short cuts through the jungle of his formulae, some of which he himself made in later years. Very few ships to-day follow Columbus' course across the Atlantic.

Let me put the matter in another way. Anyone reading the controversy between Pearson and Weldon on one side, and Bateson and his colleagues on the other, which reached its

* The author is quite aware that he has repeated himself in a way which would be unjustified had the material been put together for an article, but which is justifiable in an oration. On the other hand, he considers that it is seldom desirable to hack what was designed to be spoken into a form suitable for reading.

culmination about fifty to fifty-five years ago, might have said 'I do not know who is right, but it is certain that at least one side is wrong'. In fact both were right in essentials. The general theory of Mendelism is, I believe, correct in a broad way. But we can now see that if Mendelism were completely correct, natural selection, as Pearson understood it, could not occur. For the frequency of one gene could never increase at the expense of another, except by chance, or as we now put it, sampling errors. It is just the divergence between observed results and theoretical expectations, to which Pearson rightly drew attention, which gives Mendelian genetics their evolutionary importance.

After this preamble I pass to my main task. Pearson's connexion with this College began when he was nine years old, and was sent to University College School, where he remained for seven years. He left at sixteen and obtained a scholarship at King's College, Cambridge, at eighteen. As an undergraduate he studied mathematics and was third wrangler in 1879. He had already shown something of his future mettle by a successful refusal to attend divinity lectures. In spite, or perhaps because, of this independence of spirit he became a fellow of King's in 1880. He spent about a year in the universities of Heidelberg and Berlin, attending lectures on philosophy and Roman law as well as physics and biology. However, the most striking effect of his German year was to interest him in mediaeval and Renaissance German literature, especially the development of ideas on religion and the position of women. At about this time he began to spell his Christian name with a K instead of a C. This may have been a homage to German culture. It may have been a special homage to Karl Marx, for we know that he later lectured on Marx, and his daughter tells me that when in Germany the police once searched his rooms, and he considered that one of Marx's books was the most subversive of the documents which they found there.

In 1880 he began the study of law in London, and was called to the bar in 1881. This may have been a tribute to his father, who was a Q.C., or a means of ensuring a livelihood in future, more probably both. He also published his first books, *The New Werther** and *The Trinity, a Nineteenth Century Passion Play*. Both were anonymous, and had they been signed, would certainly have prejudiced their author's chance of appointment in many institutions, perhaps even in the Infidel College, which suffers from occasional outbreaks of respectability. For both attack Christian orthodoxy.

It was at this period of his life that he lectured on Marx to small audiences in London, on the 'Ethic of Freethought' at South Place, and to the Sunday Lecture Society on 'Matter and Soul'.

In 1884, at the age of 27, he was appointed to the Chair of Applied Mathematics and Mechanics in this College. He had only published two small papers on rather academic problems of applied mathematics. His first publication after his appointment was *The Common Sense of the Exact Sciences*, by his illustrious predecessor in this College, W. K. Clifford. His next was even more surprising. It was written in German, entitled *Die Fronica: Ein Beitrag zur Geschichte des Christusbildes in Mittelalter*, and published at Strassburg. So far as I know it was the first contribution made by a professor of this College to the history of art. It is interesting to see that he regarded this as a worthy topic of academic study. May I hope that, now that we have a Chair of this subject, our Professor may comment on Pearson's contribution to it.

He was clearly a very successful and thorough teacher of applied mathematics, mainly

* For the bibliography of Pearson's works, and for much else, I rely on E. S. Pearson's invaluable memoir (Cambridge, 1938). In this lecture I have not even mentioned some of his books.

to students of engineering. He edited de Saint-Venant's work on the theory of elasticity, and wrote the second part of Todhunter's *History of the Theory of Elasticity*. His radical activities continued. In 1885 he joined 'The men's and women's club', a small body devoted to 'the free and unreserved discussion of all matters in any way connected with the mutual position and relation of men and women'. As, in *The Ethic of Freethought*, Pearson defended the view that unmarried women should be allowed sexual freedom, it is not surprising that legends arose, and still exist, as to this club.* In fact Karl Pearson married one of its members, Miss Sharpe. To-day it is quite normal for a couple to discuss human sexual physiology before marriage. Seventy years ago it was regarded as grossly improper, and all kinds of accusations were made against those who did so. I have not the faintest doubt that in fact the male members of the club were far less promiscuous than most of their contemporaries. If to-day association with prostitutes is generally regarded as degrading, while seventy years ago it was generally condoned and not rarely approved, we owe it largely to men like Karl Pearson.

The Ethic of Freethought was published in 1888, and is a collection of lectures and essays, some of which had been reprinted as pamphlets. It is, in essence, a religious book. Pearson defined religion as 'the relation of the finite to the infinite'. 'Hence', he continued, 'all systems of religion are of necessity half truths.' The most scholarly part of the book deals with the history of religious systems, particularly in Germany. He believed that such a study was part of the duty of an educated man or woman. I read a few sentences. 'By studying the past I do not mean reading a popular historical work, but taking a hundred, or better fifty, years in the life of a nation, and studying thoroughly that period. Each one of us is capable of such a study, though it may require the leisure moments, not of weeks, but of years. It means understanding, not only the politics of that nation during those years; not only what its thinkers wrote; not only how the educated classes thought and lived; but in addition how the mass of the folk struggled, and what aroused their feeling and stirred them to action. In this latter respect more may be learnt from folk-songs and broadsheets than from a whole round of foreign campaigns.'

The book is largely a record of its author's search for truth among religious systems. One chapter is devoted to the mystic Eckehart, and was the first introduction of that remarkable thinker to the British public. Of all the systems examined there can be no doubt that that of Spinoza appealed most deeply to Pearson; and he devoted another chapter to demonstrating Spinoza's debt to Maimonides. If I may be allowed to express a regret which is in no sense a criticism, it is that Pearson's acquaintance with Indian philosophy was confined to translations of Hinayana Buddhist scriptures. I think that he would have recognized more kindred spirits in such ancient Hindu thinkers as Yajnavalkya and the great anonymous humanist whose words are preserved in the first section of the Bṛhadaranyaka Upanishad.

It is a little surprising that the title page does not mention the author's professorship at University College. Perhaps his senior colleagues thought that such a mention would have got him into trouble.

If, in 1890, one had had to pass judgement on Pearson, it might have run as follows,

* [As might be expected, the critics tended to fix on only one side of the picture of the ideal relationship between the sexes in a socialist state which Pearson elaborated in his lectures on "The Woman's Question" (1885) and "Socialism and Sex" (1886), afterwards published in *The Ethic of Freethought*. Ed.]

'He is a first-rate teacher of applied mathematics, and a scholarly compiler of the work of more original men. He has a knowledge of literature and art most unusual in a professor of mathematics. He is somewhat of a radical, but he is only thirty-three years old. He will settle down as a respectable and useful member of society, and may expect a knighthood if he survives to sixty. He will never produce work of great originality, but the College need not be sorry to have appointed him.' Had this judgement been correct, we should not be here to-day.

In 1890 two events occurred which, in my opinion, shaped the course of Pearson's future life. He applied for, and received, the lectureship in Geometry at Gresham College; and W. F. R. Weldon succeeded Lankester in the chair of zoology at University College. At Gresham College he could lecture on what he pleased. His first set of lectures developed into *The Grammar of Science*, his main contribution to philosophy. Later series dealt with 'The Geometry of Statistics', and 'The Laws of Chance'. But since the discussion of probability and statistical method in the first edition of *The Grammar of Science* is superficial, we may take it that in 1891 he had not considered the subject seriously. He certainly did so in later years. I have little doubt that the stimulus to do so came largely from Weldon.

The Grammar of Science is a very remarkable book. Pearson claimed that material objects were merely a conceptual shorthand used to describe regularities in our sense-impressions. This idea is hard to develop, if only because our language is in terms of material objects such as eyes and brains. He did not in fact develop it without some self-contradiction, at least on the verbal level. But he did so, in my opinion, with much less self-contradiction than contemporaries such as Mach and Avenarius. He must be regarded as one of the founders of the important school of logical positivism.

I can well remember the impression which his book made on me when I first read it about 1909. If it is less impressive to modern readers, this is probably because physical theories have changed profoundly, a fact which would in no way have surprised or distressed its author. I do not personally think that Pearson's philosophical views are correct. Nevertheless, a man who first states an important doctrine clearly, even if it is subsequently rejected, is a moment in the thought process of humanity. We can best see whether Pearson did this by listening to the judgement of one of his adversaries. In 1908 Vladimir Ilyitch Lenin wrote *Materialism and Empirio-criticism*. This was an attack on people who, in his words, or rather those of his translator, 'under the guise of Marxism were offering something incredibly muddled, confused and reactionary'.

Now Lenin disagreed strongly with Pearson, and claimed, in my opinion correctly, to have found self-contradictions in his arguments. Nevertheless, he found him vastly clearer than other Machians. Let me read a few of Lenin's sentences. 'The philosophy of Pearson, as we shall repeatedly find, excels that of Mach in integrity and consistency' (p. 119).* 'The Englishman, Karl Pearson, expresses himself with characteristic precision, "Man is the creator of natural law".' (p. 221). And finally (p. 243) Lenin described him as 'This conscientious and scrupulous foe of materialism'. Unfortunately, I do not know how precise is this translation from the original Russian. But praise of this kind from an opponent is in my opinion worth a great deal more than either the assent of uncritical disciples or the patronizing acknowledgements of successors who claim to have improved on Pearson's

* The references are to the pagination in Vol. 11 of Lenin's *Selected Works*. London, Lawrence and Wishart, 1939.

treatment of the subject. The only other contemporary British opponent of materialism to whom Lenin was equally polite was James Ward. I cannot help thinking of Dante's treatment of Saladin, who was, of course, in hell, but so far from suffering from heat, cold, or other torments, was housed in a noble castle. Whatever may be the fate of Pearson's philosophy in his own country, *The Grammar of Science* is assured of attentive reading in those states where Leninism is orthodox.

To go back to Pearson's own views, I quote three sentences from the *Grammar*, which I think illustrate the strength and the weakness of Pearson's approach to science. 'The unity of all science consists alone in its method not in its material.' 'No physicist ever saw or felt an individual atom. Atom and molecule are intellectual conceptions by aid of which physicists classify phenomena and formulate the relationship between their sequences.' The strength is shown by the fact that the distributions, which Pearson worked out to describe Weldon's measurements of populations of crabs, will equally well serve to describe populations of stars, manufactured goods, durations of life, incomes, barometer readings, and so on. The weakness is shown by the fact that physicists have, during this century, seen individual atoms, or rather atomic nuclei, by the tracks which they make when moving rapidly. Pearson's philosophy discouraged him from looking too far behind phenomena. It was, I think, for this reason, that he never accepted Mendelian genetics, although the *Treasury of Human Inheritance* and his own monograph on albinism contain plenty of evidence in its favour.

His later series of Gresham Lectures dealt with statistics and probability, particularly with graphical methods of representing distributions. I have no doubt that they were written partly as a result of the questions which Weldon began putting to him soon after his arrival in University College. But his full answer to these questions is to be found in the great series of memoirs on the Mathematical Theory of Evolution which were published in the *Philosophical Transactions* of the Royal Society between 1893 and 1900. It is not too much to say that the subsequent developments of mathematical statistics are largely based on Pearson's work between 1893 and 1903. Perhaps we shall be helped to estimate its importance by an exercise in hypothetics. What would have been the effect on Pearson had Bateson obtained the Jodrell Chair of Zoology in place of Weldon? And what would have been the effect had our College contained an economist or engineer interested in what is now called Quality Control? Although Bateson was as interested as Weldon in animal variation, he was more concerned with exceptions, and with discontinuous, or as Pearson and Lee called it in 1899, exclusive inheritance. I doubt if Bateson would have put his questions in a form which would have aroused Pearson's interest. If he had done so, they would probably have discovered what is now called Mendelism. For Bateson, before reading Mendel's paper, did not realize the necessity of dealing with large samples, which Pearson certainly did.

If an economist or technologist had interested him in the variation of manufactured goods, he would have had to deal, as he did, with skew variation. He would presumably have used correlation to measure the likeness between the products of the same craftsman or machine as he in fact used it to measure the likeness between the children of the same parents. Perhaps in 1901 he might have founded a journal *Technometrika* not wholly unlike *Biometrika*. He would almost certainly have invented some, at least, of the statistical methods now used in industrial quality control. He might perhaps have added 1 % or so to the industrial productivity of Britain in the early years of this century.

The papers to which I refer are hard to read because Pearson reached his conclusions by algebraical and arithmetical methods which are now seen to be needlessly laborious. Many of them have since been simplified. As a humble tribute to Pearson I have, as I believe, simplified the first of them, which deals with the dissection of a skew frequency distribution into two normal distributions. By an elementary transformation I have thrown his rather formidable nonic equation into a form which allows numerical tabulation, and this tabulation is now under weigh in the electronic laboratory of the Indian Statistical Institute. I hope that as a result, the method will be available to statisticians less pertinacious than Karl Pearson.

Commenting on a particular passage in (I think) one of Beethoven's works, a German musical critic remarked 'Hier ist Titanenthum Pflicht'. (Here titanicity is a duty). Karl Pearson attacked Olympus by piling Ossa on Pelion rather than by seeking an easy path. If we, his successors, have made statistical theory relatively easy, and much of Pearson's mathematics are no longer used, we should remember that we are treading in the footsteps of an intellectual titan.

The germs of many later developments in mathematical statistics are to be found in these papers. Thus, in Contribution III to the theory of evolution, Pearson discussed 'the best value of the correlation coefficient' based on a given sample. He decided on the value which maximized the chance of obtaining the observed sample. This method was developed by Edgeworth as 'the method of maximum credibility', and by Fisher as 'the method of maximum livelihood'. In the succeeding paper, with Filon, Pearson developed it further. Critics have asked why he did not generalize it. I think one possible answer is as follows. The expression 'the best' is unfortunately seldom applicable to statistical estimates. The best for one purpose is not usually the best for another. I think Pearson realized this. Some of his successors have not.

In 1900 Pearson attacked the problem of curve fitting. Having fitted the best available curve to a series of data, for example, the numbers of human beings whose heights were in intervals such as 70–71 in., he asked what was the probability that a sample from a population truly represented by his curve should fit it as badly as, or worse than, the sample in question. If the chance was 1 in 3, there is no good reason to doubt the validity of the theory on which the curve is based. If the chance was 1 in 300, the theory is almost certainly wrong, though it may be a useful approximation for some purposes. But the question arises 'what is a bad fit?' Is 38 a worse fit to an expected number of 30 than 4 to an expected number of 10? (It is not!) And how are we to combine these in an overall estimate of badness of fit? Pearson solved this problem by the invention of the function of observations called χ^2, which increases as the fit becomes worse.

This has turned out to be an immensely powerful tool, and is used on a huge scale. To take one example, in the last number of the *Journal of Genetics*, at least fifty-three values of χ^2 were calculated by three different authors. But now comes the curious and characteristic fact. None of these authors used χ^2 as a test of curve-fitting, and it is very rarely so used. It is used as a test of agreement with hypothesis wherever the hypothesis is tested by counting individuals. And it is used, as Pearson pointed out that it might be used, to discover whether a number of sets of data agree with the same unknown hypothesis. For example, if the total of a number of families contains about 17 % of a particular type we may have had no reason beforehand to expect 17 % rather than any other frequency. But we can use χ^2 to determine whether some of the families have a proportion which diverges

more from 17 % than could reasonably be accounted for by chance. If not, but only if not, we can justifiably pool the data. In this case χ^2 is said to be used as a test of homogeneity.

This is, of course, a commonplace with great human achievements. The wheel was invented for use in chariots, carts, and so on. But to-day most wheels are used, not for the support of vehicles, but for power transmission. Perhaps the majority of wheels in England are inside watches. It was absolutely characteristic of Karl Pearson that his intellectual inventions were often extremely general. He obtained a solution of a problem which was of such generality that it had entirely unexpected applications.

But for this very reason it often had a limited applicability to the problem for which it was originally designed. In the last few years many experiments have been done on artificial selection of quantitative characters, particularly in *Drosophila* and mice. Their results during the first few generations are often much as Pearson would have expected. But after this they diverge very greatly. In spite of this they are best described by the use of the mathematical tools which Pearson first applied to such problems, that is to say by describing changes in the moments of character distributions, and simple functions of them such as standard deviations and correlations. One can only defeat Pearson intellectually with the weapons which he himself forged. If I may be allowed to quote William Blake,* Pearson's main service to humanity was

> In all his ancient strength to form the golden armour of science
> For intellectual war.

About the same time he began not only to use data collected by Galton and others, on man and other animals, but to collect his own. Among the important biological results of this period were the demonstration that fertility is inherited both in our own species and in race-horses. As an example of his thoroughness I mention his measurements of the same human bones after various periods of wetting and drying, which never changed their length by as much as 1 %, though they did change it.

It was probably through Weldon that he came to know Galton. This very remarkable man had, among other things, invented the recognition of criminals by finger prints, and psychoanalysis (as may be seen from pages 185–207 of *Inquiries into Human Faculty*). Much of Pearson's work in the '90's was a development of the notions used by Galton in his *Natural Inheritance* in 1889. However, there is no reason to think that Pearson's one serious excursion into practical psychology owed anything to Galton. This is described in a pair of papers published in 1899 and 1902 alleged to be on the mathematical theory of errors of judgement, but in fact incorporating a series of measurements made on the same material by Dr Alice Lee, Dr Udny Yule, and Pearson, and by Lee, Dr Macdonell, and Pearson. Each observer had, of course, a characteristic bias and a characteristic spread round the mean. But what was utterly unexpected was the discovery that the errors made by two observers varied in Pearson's words, sympathetically. In fact in one series, Lee and Macdonell showed a high correlation, Pearson being independent. He attributed this to 'the influence of the immediate atmosphere'. Others might have attributed it to telepathy.

Some of his finest work at this time was with Lee, Bramley-Moore and Beeton on the inheritance of human fertility and longevity. I cannot say more for the value of this work than that I could find no better data on which to base a theory which I published in 1949,

* *Vala*, or *The Four Zoas* (End of last Night).

and which I venture to think explains some results which Pearson found surprising at the time, though, of course, he published them, and did his best to explain them.

In 1901 the first volume of *Biometrika* was published, partly no doubt because the Royal Society, although it had awarded him its fellowship in 1896 and its Darwin medal in 1898, objected to publishing advances in mathematical and biological knowledge in the same paper! *Biometrika* has not fulfilled what Galton, in its first number (p. 9) stated to be the primary object of biometry, namely 'the discovery of incipient changes in evolution which are too small to be otherwise apparent'. The reason for this failure is simple. The mean rate of increase in tooth length during the evolution of the horse since the Eocene is now known to have been about 4 % per million years. Such evolution could not be detected in a human lifetime. But the aims stated in the editorial introduction, presumably the joint work of Pearson, Weldon and Davenport, were fulfilled. In particular the first number contained a paper by Weldon on variation in snails whose importance he did not live to realize. He found that natural selection in a snail species weeded out extremes, reducing the standard deviation of a metrical character without affecting the mean. We now know that this centripetal selection is very common. Had Weldon lived longer he would presumably have discovered this, and the whole history of biometry would have been very different.

In 1903 Pearson's Department received a grant of £500 from the Drapers' Company, and these grants, at the rate of £500 per year, continued till 1932. In 1903 this sum was worth about £3000 or more to-day, and went partly in the payment of Dr Lee and other computers, partly for instruments, and partly for printing.

I have no idea how Pearson obtained this money. We may be sure that he did not either flatter rich men or promise to improve the national health and intelligence in their lifetimes. Perhaps Galton had the ear of some rich acquaintances. Perhaps too, at that time our ruling classes were less permeated than now with the ferocious contempt for the pursuit of knowledge for its own sake, which is voiced in the Archbishop of Canterbury's sermon of March 24, 1957. To-day it is not hard to get money for research which may have economic, military or hygienic advantages. It is extremely hard to do so for the mere search for truth.

About this time Pearson began the series of papers on human biology for which he is best known in some quarters, and the majority of which, I think, were joint work. Even where his name did not appear on papers, I think our chairman will agree that nothing was published from his laboratory without his *imprimatur*, and some of such work must at least briefly be considered here.

Many of these papers are as fresh to-day as when they were written. To take an example, in my opinion nothing since written on human craniometry has in anyway superseded Pearson's and Davin's great memoir of 1924. Some of this work was, at the time, of inestimable value. I think particularly of the *Treasury of Human Inheritance*. This is still indispensable. Nevertheless, we now know that it is possible to distinguish between conditions (for example, haemophilia, Christmas' disease, Owren's disease, and so on) which were inevitably classed under a single category by the writers of the *Treasury*. The more polemical writings of this period are of less value to-day, as Pearson doubtless realized when he called a series 'Questions of the Day and of the Fray'. The Fray in question was a many-sided contest. On the one hand, Pearson and his colleagues attacked those who underestimated the importance of heredity, including those who exaggerated the harm done by parental alcoholism. But they also attacked those who oversimplified it, including many Mendelians and many eugenists. Other attacks were on statistical data alleged to

prove the value of immunization to diseases. These attacks were fully justified. Mental defect is certainly not a Mendelian character. As Pearson and Jaederholm showed, the distribution of intelligence quotients in defectives is the tail of a nearly normal frequency distribution. Forty-three years later we can say a great deal more about it. We can say, for example, that phenylketonuria is a chemically definable character inherited as a Mendelian recessive, and accounting for perhaps 1 % of certifiable mental defect. But the mental defect of phenylketonurics is graded, and a few of them are stupid, but not sufficiently so to be classed as feeble-minded. In fact the diagnosis of phenylketonuria enabled Penrose to dissect the distribution of human intelligence quotients into two very different but still overlapping distributions. The notion that such a dissection is possible was Pearson's first contribution to biometry.

Again, one series of memoirs was entitled *Studies in national deterioration*. This is a polemical title. And it is a fact that as regards most measurable characters the nation has not deteriorated. It may have done so as regards its 'nature' or inborn capacities. I think that if Weldon had lived Pearson would have realized the ubiquity of centripetal selection, and that in fact both the most successful and the least successful members of society were breeding more slowly than those a little below the median. It is, however, easy to be wise after the event. Moreover, Pearson and his colleagues were completely right in one respect. Even if, in spite of his predictions, the nation has improved in some measurable directions, it would have improved more if, say, a million children who were born to unskilled labourers had been born to skilled workers, teachers, and the like.

No such criticism is possible of the mathematical tables which he edited, and in whose computation he played a large part. Their utility was well shown by the fact that 'pirated' editions of them were soon published in America. The subsequent development of statistics is largely based on them. Even the advent of the electronic computer has not yet superseded them. They were published from 1914 to 1934, and the *Tables of the Incomplete Beta-function*, published in Pearson's seventy-eighth year, were his last, but not his least, contribution to science. It appears that no one has yet discovered how to use an electronic computer as efficiently as Pearson used his teams of devoted, painstaking, and remarkably accurate, lady assistants.

In 1911 Galton died, and left funds for the endowment of a Chair of Eugenics, of which Pearson became the first occupant. At last he was able to give up the teaching of applied mathematics to engineers and physicists, and in the next year the present laboratory was begun. Fortunately it was completed by 1914 though it was commandeered as an annexe to the hospital, and he did not get into it till after the war. From 1914 till 1919 he did very little but war work, first for the Board of Trade and later on calculation of trajectories of artillery. When in 1920 the Department of Applied Statistics was formally opened, he was sixty-three years old, and he had, among other things, to develop a new course of lectures and practical work. In 1923 he began a series of papers which combined biometrical and historical research. He was able to measure the skulls of a number of distinguished men and compare them with contemporary portraits. In the course of this work he played the detective, and reconstructed the murder of Lord Darnley, second husband of Queen Mary of Scotland; and his comments on the history of the Reformation in Scotland are well worth reading. To the same period belongs his great life of Galton, which involved much historical research.

I have devoted this lecture entirely to Pearson's published works. If he could hear it I

believe his main criticism would be that I have said too little about his fellow workers; for much of his work was in collaboration. He had a wonderful gift for inspiring loyalty in his colleagues, of which more will be said to-day by others. I myself only met him frequently in the last years of his life, and can merely say that he was most gracious to me, though my outlook on many biological questions was very different from his own. He resigned his Chair in 1933, but published one book and at least three scientific papers before his death in 1936.

It remains to say a few words about what Samuel Butler would have called his life after death, the results which are still accruing from his original thought. To begin with, all subsequent statistical work is based on the foundations which he laid. If we sometimes find it more convenient to speak of a variance ratio where he would have used a coefficient of correlation this does not mean that his work on homotyposis is obsolete. If his system of frequency distributions is less used for biological data than he might have hoped, yet Gosset, Fisher and others have found that they describe exactly the distributions of many statistical estimates based on finite samples.

At University College his work is being carried on by three professors.* Under his son the Department of Statistics has become the leading teaching department in that subject in Britain, and the new Biometrika Tables, to take only one example of its work, continue his father's great tradition. Prof. Fisher, who succeeded him as Professor of Eugenics, did very great services to statistics in simplifying and rendering more accurate a number of statistical procedures; and, by the application of methods which owed much to Pearson's great memoir on homotyposis, made agricultural experiment an exact art. To mention only one of Fisher's contributions to eugenics, he established a laboratory for human serology, and his interpretation of the human *Rh* antigens has had a considerable influence on the prevention and cure of a very serious human congenital disease. His controversies with his predecessor were perhaps inevitable between two men each so determined to defend the truth as he saw it. Under Prof. Penrose the Department has swung back towards Pearsonian methods. If I may mention two researches which I believe would have delighted Pearson particularly they are the work of Karn and Penrose on infantile mortality as a function of birth weight, which measured natural selection in man with an accuracy which he would have envied, and that of Penrose on abnormality in the offspring as a function of parental age and maternal parity, a subject which Pearson had broached in his 1914 memoir on the handicapping of the first-born.

My own department of biometry has not been so fortunate. I was Professor of Genetics till 1937. I should not have accepted the Weldon Chair had I not been promised accommodation for Biometrical work. Owing to the war, and for other reasons, this promise was not kept. I have been unable to carry out the duties of this chair adequately. In my opinion the best biometric work of the last twenty years has been carried out by Teissier and Schreider in France, and by Mahalanobis and his colleagues in India. In a few months in India I have been able to start new lines of biometrical research. I think particularly of the work of S. K. Roy, now I hope in press, in which he took up the problem of homotyposis where Pearson left it in 1903. A study of some 60,000 flowers from three different plants (to be compared with Pearson's 4443 capsules from 176 poppy plants) has shown that individual plants not only have their characteristic means, but their characteristic standard devia-

* Perhaps I should also include the Professors of Astronomy and Art History, and those of Engineering.

tions, and that both of these alter in a characteristic manner during a season. I believe that the opportunities for Biometric research are now better in India than in Britain, and for this reason among others I have thought it my duty to migrate there. To quote Karl Pearson's most loved poet

In Vishnu-land what avatar?

Whatever the fate of Pearsonian biometry in Britain, I believe that it will live and flower in India.

To me at least there seems to be an element of hypocrisy about the present celebrations. I believe that we should be honouring Pearson more effectively if, to take one example out of many possible, we ensured that the College Library possessed copies of all his works, placed where students could consult them, than by making speeches and eating food. I mention this particular example as I have been trying, without the faintest success, to secure such accessibility for at least ten years.

Pearson's work for free thought and the emancipation of women has been successful in this country, if not always as quickly so as he hoped seventy years ago. The history of art is now taught in this College. His work for socialism has not been as successful here as he hoped. Nor would he have approved of many features of the socialistic systems of the Soviet Union and China. Here again I believe his real heirs are to be found in India, where the editor of *Sankhyā*, the Indian Journal of Statistics, is also the principal planner of the approach to socialism under the second Five-Year Plan.

I fully realize that I have not done justice to my subject. The task set me was an impossible one. No one man now alive could do justice to the breadth of Karl Pearson's interests and achievements. But I thank you for joining with me in celebrating the memory of this great man.

AN ANALYSIS OF THE DATA FOR SOME EXPERIMENTS CARRIED OUT BY GAUSE WITH POPULATIONS OF THE PROTOZOA, *PARAMECIUM AURELIA* AND *PARAMECIUM CAUDATUM*

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1. Introduction

In his book, *The Struggle for Existence*, Gause (1934, pp. 96 *et seq.*) describes a very interesting set of experiments which he carried out with populations of the Protozoa *Paramecium aurelia* and *P. caudatum*. The growth in numbers of these populations was observed when each of the species was living alone, and also when both were living together in a constant volume of nutrient medium. This consisted of 5·0 c.c. of Osterhout's balanced physiological salt solution in which was suspended a standard measured amount of a freshly grown culture of *Bacillus pyocyaneus*, the latter serving as a food supply for the Protozoa. This medium was freshly prepared each day, and it is stated that in this solution no multiplication of the suspended bacteria takes place. Each set of experiments was replicated either three or four times, and the cultures were kept in a moist thermostat at 26° C. At the end of every 24 hr. a sample was taken from each culture in order to count the number of individuals present; and after centrifuging down the remaining individuals, the old medium was removed and replaced with the same volume of a freshly prepared suspension of bacteria.

This series of data is therefore of some interest from the point of view of population mathematics. For essentially, what we are given here are the changes in the numbers of two species, either living alone or competing together in an environment of fixed size, and under as constant a set of physical conditions and amount of food as the experimental procedure admitted. The two species observed are of a comparatively simple biological type and, under these experimental conditions, can be regarded as multiplying merely by division of the individuals. This greatly simplifies any mathematical model we adopt, since the complications due to age-specific birth and death rates and the age structure of the population can be neglected. Moreover, since each set of experiments was replicated, we are able to estimate, in addition to the mean values of these stochastic processes, the degree of variance between replicates at some time, or over some particular period in the development of the populations. In the case of these data, however, any estimate of this inter-replicate variance will have at least two components. There is, in the first place, the variance which is to be expected between a set of comparable stochastic processes at some given time; and, in addition, there is the composite variance due to the sampling of each population as a whole, and to the method of counting the individuals in the samples.

The data for the individual replicates of *P. aurelia* and *P. caudatum*, when each was living alone, and of the mixed cultures, are given by Gause in the appendix to his book (1934, Table 3, pp. 144–5). (It should be noted that each entry in this table must be multiplied by a factor of ten in order to find the estimated total numbers in each culture as a whole.) Each population of a single species was started with 20 individuals (or 40 in the case of the

mixed cultures) and the total numbers thereafter increased until there might be around 5000–6000 individuals in the case of some cultures. There was, however, a considerable degree of variation between the replicates at each of the successive sample censuses.

Evidently we are dealing here with a set of experimental populations in which the numbers are not particularly small. We might assume, therefore, as a first approximation that the changes in the mean values of these stochastic processes could be described adequately in terms of a simple deterministic model; and the object of the present paper is to examine these data in the light of this hypothesis.

2. MATHEMATICAL MODEL

The mathematical model which will be used in the analysis of these data is the familiar one associated with the names of Lotka (1925, 1932) and Volterra (1926, 1931). Suppose we have two competing species of living organisms, S_1 and S_2, whose populations consist respectively of N_1 and N_2 individuals at any given moment. If either of these species were living alone, under some constant set of physical conditions, such as the temperature, relative humidity, the hydrogen ion concentration of the medium, and so forth and no limitations whatsoever were placed upon its increase in numbers; and if the amount of some particular source of food supply is unlimited; then we assume that the species would increase in numbers at a rate defined by

$$\frac{dN}{dt} = rN = (b-d)N,$$

where r is the intrinsic rate of increase, representing the difference between the 'birth-rate', b, and 'death-rate', d, of the species under the given conditions. In a finite environment, however, when some source of food supply is assumed to remain constant in amount, an approach to this intrinsic rate of increase would only be made when the number of individuals is relatively small, and we suppose that any increase in numbers will tend to have an adverse effect on the relative rate of increase of the population. Hence, when limitations either of space, and/or food supply, are placed upon the increase in numbers, we have in general for a particular species living alone

$$\frac{dN}{N\,dt} = F(N),$$

where the function F is subject to the condition,

$$\frac{dF}{dN} < 0.$$

As a first approximation, the simplest function which fulfils these conditions is evidently

$$F(N) = r - aN,$$

where a is a positive parameter; so that we have the well-known logistic differential equation

$$\frac{dN}{dt} = (r-aN)\,N,$$

whence, by integration,

$$N_t = \frac{K}{1+Ce^{-rt}},$$

where $K = r/a$, and C is a constant defining the initial state of the system.

When the two species S_1 and S_2 are competing together in a limited environment for some common food supply remaining constant in amount, we suppose that any increase in numbers of the species S_1 will tend to have an adverse effect, not only on its own relative rate of increase in numbers, but also on that of the species S_2; and, similarly any increase in numbers of S_2 will have an adverse effect both on its own relative rate of increase and on that of the species S_1. We have, therefore, the perfectly general equations describing the interaction between two competing species, under some constant set of physical and environmental conditions,

$$\frac{dN_1}{N_1 dt} = F_1(N_1, N_2),$$

$$\frac{dN_2}{N_2 dt} = F_2(N_1, N_2),$$

where the functions F_1 and F_2 are subject to the conditions,

$$\left.\begin{matrix} \dfrac{\partial F_1}{\partial N_1} & \dfrac{\partial F_1}{\partial N_2} \\ \dfrac{\partial F_2}{\partial N_2} & \dfrac{\partial F_2}{\partial N_1} \end{matrix}\right\} < 0.$$

For our present purpose it is not necessary to discuss the properties of this general system of equations. Clearly, by an extension of the argument used above for the case of a species living alone, the simplest forms of the functions F_1 and F_2 which fulfil these conditions, is a set linear in the variables N_1 and N_2. We write, therefore,

$$\frac{dN_1}{dt} = (r_1 - a_1 N_1 - b_1 N_2) N_1,$$

$$\frac{dN_2}{dt} = (r_2 - a_2 N_2 - b_2 N_1) N_2,$$

where r_1 and a_1 are the logistic parameters for the species S_1, if it were living alone, and similarly r_2 and a_2 those for the species S_2; while the positive parameters b_1 and b_2 measure the degree to which each species affects the relative growth rate of the other. In general, all these six parameters are unequal, in which case there is no explicit form of solution of this set of differential equations; but, when they are not all unequal, various special cases may arise, one of which appears to be applicable to the present series of data.

3. Application of the models to the present data

In applying this model to the results of Gause's experiments, there is, however, one important point to be borne in mind. Hitherto, in developing this model, we have supposed that the species S_1 and S_2 were living, either alone or together, in some ideally isolated system, abstracted from the rest of nature, and that the activities of the observer had no effect on the growth in numbers of the populations. But, in these experiments, every day before the nutrient medium was changed, each culture was carefully stirred and a sample of 0·5 c.c. (one-tenth of the total volume) was removed, and the number of individuals in it was counted. Having been counted, this sample was destroyed (Gause, 1934, p. 97). Thus, the activities of the observer, in this case, were equivalent to those of an intermittent predator on the populations of *Paramecia*, resulting in the sudden destruction at the end

of every 24 hr. of—on the average—one-tenth of the total number of individuals present at that time.

If we define n_t as the number of individuals counted in the sample of 0·5 c.c., then the estimated total number of individuals in each culture, at the time it was removed from the incubator, is

$$N_t = 10n_t.$$

After renewal of the medium, each culture when it was returned to the incubator therefore on the average contained, at the beginning of the interval t to $t+1$,

$$N'_t = 0{\cdot}9N_t$$

individuals. We assume, in the case of each species when living alone, that during this interval t to $t+1$, the population was growing in numbers according to the logistic differential equation

$$\frac{dN}{dt} = (r - aN)\,N, \tag{3·1}$$

and we require estimates of the parameters r and a, given the values of n_t at successive intervals of time.

From (3·1) we have by integration

$$N_t = \frac{K}{1 + C\,e^{-rt}} \quad (K = r/a); \tag{3·2}$$

whence

$$(K - N_t)/N_t = C\,e^{-rt}; \tag{3·3}$$

and at time $t+1$

$$N_{t+1} = \frac{K}{1 + C\,e^{-r(t+1)}}.$$

Substituting (3·3) in this last equation, and writing

$$\lambda = e^r,$$

we have, after a little rearrangement, the following equation relating the number of individuals at the end of the interval t to $t+1$ with the number at the beginning,

$$N_{t+1} = \frac{\lambda N_t}{1 + \alpha N_t},$$

where

$$\alpha = (\lambda - 1)/K.$$

Since we have defined the number of individuals at the beginning of each interval as $N'_t = 0{\cdot}9N_t$, owing to the destruction of a tenth of the population in the sample discarded after counting, we have for the successive intervals,

$$N_{t+1} = \frac{\lambda N'_t}{1 + \alpha N'_t}, \tag{3·4}$$

where N'_0 is the number of individuals with which the cultures were started. From (3·4) we have

$$\frac{N'_t}{N_{t+1}} = \frac{1}{\lambda} + \frac{\alpha}{\lambda} N'_t, \tag{3·5}$$

a linear relationship from which λ and α can be determined, and hence the parameters r and $K = r/a$ of the logistic equation.

It will be noted that (3·4) can also be written as

$$N_{t+1} = \frac{\lambda' N_t}{1 + \alpha' N_t},$$

where $\lambda' = 0{\cdot}9\lambda$ and $\alpha' = 0{\cdot}9\alpha$; and $N_0 = N_0'/0{\cdot}9$. Thus, since $N_t = 10n_t$, if we work in terms of the numbers recorded by Gause, each multiplied by a factor of ten (although this is not really necessary), the estimated total numbers of individuals in the population, allowing for the activities of the observer, also follow a logistic curve defined by

$$N_t = \frac{K'}{1+C'e^{-r't}} \quad (K' = r'/a'), \tag{3·6}$$

provided we adjust the initial numbers by dividing them by a factor of 0·9. The relationship between K', r' and the K, r of (3·2) is given by

$$r' = r + \log_e(0{\cdot}9), \quad K' = \frac{(0{\cdot}9e^r - 1)K}{0{\cdot}9(e^r - 1)}.$$

Thus, in the case of these experiments, there are, when each species of *Paramecium* is living alone, two logistic equations to be considered. There is, in the first place, equation (3·2) which in terms of the model describes the expected average change in numbers of the population when no activities of the observer affect it in any way; and secondly, we have equation (3·6) describing the interaction between the population and the observer, in so far as the latter is acting as a predator on the numbers of the former. We may work in terms of either of these equations, transforming the estimated values of the parameters from the one into the other whenever necessary.

4. Experiments with each species of *Paramecium* living alone

The data for these experiments are given by Gause (1934, Table 3, pp. 144–5), in the form of the number of individuals per 0·5 c.c. of culture. Each entry in this table thus represents, presumably, the result of counting the number of individuals in the sample of 0·5 c.c. removed each day from a total volume of 5·0 c.c. It is, however, not at all clear whether the entire sample was always counted, since the occurrence of either zero or five in the last figure of the more dense populations suggests that some form of subsampling was employed in doing the actual counts, more particularly when the populations were approaching their upper asymptote in numbers. It appears from this table that the first sample census was taken when each replicate was 2 days old, and that thereafter each surviving replicate was sampled daily until the nineteenth day after the start of the experiments. From day 20 until day 25, when the series ends, there are no records for the individual replicates, but it is stated in the text (p. 99): 'A separate counting of the number of individuals in every culture was discontinued from the twentieth day, and we began to take average samples from similar cultures.' Working from an origin of time, therefore, we have the following information for making any estimates of the variance between replicates.

Type of culture	No. of replicates	Days
P. aurelia alone	3	2–19
P. caudatum alone	4	2–13
	3	14–19

The values of the parameters in the logistic equations for the two species were estimated from the mean values ($\overline{n}_t$) of each set of replicates, as given by Gause in his table, by means of the equation (3·5), viz.

$$\frac{N'_t}{N_{t+1}} = \frac{1}{\lambda} + \frac{\alpha}{\lambda} N'_t,$$

where $N_t = 10\overline{n}_t$ and $N'_t = 9\overline{n}_t$ for $t = 2, 3, 4, \ldots, 25$. No census apparently was taken at $t = 1$; but initially $N'_0 = 20$ in both cases, and in order to retain this information in the calculations, the ratio $\sqrt{(20/N_2)}$ was taken as an estimate of N'_0/N_1. This is quite a close approximation when N'_0 and N_2 are small compared with the upper asymptote in numbers. Writing

$$N'_t/N_{t+1} = y, \quad N'_t = x,$$

we have the linear relationships between these variables

$$(y - \overline{y}) = b(x - \overline{x}).$$

There is here a question as to the best method of estimating the parameter $b = \alpha/\lambda$. We are not regarding, in the usual way, y as a dependent and x as an independent variable, for which we require an estimate of the regression coefficient b of y on x. Moreover, both these variables are subject to error. Actually, in the present instances, a method suggested by Rhodes (1940) for estimating the parameters of a logistic by means of a very similar type of linear relationship was adopted, and the value of the parameter b was taken to be

$$b = \left\{\frac{\Sigma(y - \overline{y})^2}{\Sigma(x - \overline{x})^2}\right\}^{\frac{1}{2}},$$

using the entire data recorded by Gause from day 0 until day 25. As a result the following estimates of the parameters λ and α were obtained.

	λ	α
P. aurelia	2·4905	0·00026195
P. caudatum	2·2042	0·00058189

These numerical values of the parameters λ and α were then inserted in equation (3·4), and starting at $t = 0$ with $N'_0 = 20$ individuals in each case, the expected total number of individuals at time t, and hence of the number in the sample removed and discarded at that time, were calculated by means of a repeated application of the equation. (Since no sample was taken at $t = 1$, we have $N'_1 = N_1$ in both cases.) The expected mean number of individuals ($\overline{n}_t$) in the sample of 0·5 c.c. are given in Table 1, together with the observed mean numbers from $t = 2$ to $t = 19$. The calculations were stopped at this point because this was the last day for which the data for the individual replicates were recorded, and also because it was evident that the upper asymptote in numbers for the theoretical curves was attained by this time.

A test of the 'goodness-of-fit' of these calculated curves can be made by comparing the pooled inter-replicate variance between days 2 and 19 with the mean square deviation between expected and observed in Table 1. There are, in each case, eighteen observations

from which the latter can be calculated, and the inter-replicate variances can be based, respectively, on 36 and 48 degrees of freedom. If we were to apply the usual variance-ratio test of significance, then taking as the number of degrees of freedom, $n_1 = 18$ and $n_2 = 36$ or 48, the 5 % points of F would be approximately 1·89 and 1·81. But, it is not at all clear whether we would be justified in applying this test to the present data. Although the successive daily samples withdrawn from each population are independent, the number of individuals in a particular population observed over a period of time are likely to be serially correlated. A replicate, for instance, which is greater than the mean at some given time, is likely also to be greater than the mean on the next day. Leaving aside this point, however, a rough test of the goodness-of-fit in the present examples would be to regard as unsatisfactory any ratio between these variances which was greater than two.

Because the magnitude of the figures recorded for the individual replicates differed quite considerably, not only between themselves at some given time, but also as the populations increased in numbers, the calculations were carried out in terms of the logarithms (to the base 10) of the numbers observed. Estimating the pooled inter-replicate variance in the usual way, by eliminating the variance between days, the following values of s^2 were obtained for the period of time 2–19;*

	D.F.	s^2
P. aurelia	36	0·005825
P. caudatum	48	0·01657

Thus, since in Table 1 the mean numbers of *P. aurelia* are each based on three replicates, we should expect the mean variance between $\log_{10}$ expected and $\log_{10}$ observed to be around $s^2 = 0{\cdot}005825/3 = 0{\cdot}00194$, if the calculated curve is a good fit to that observed. In the case of *P. caudatum*, the first twelve observations are based on four replicates, and the remainder on three replicates, and thus we should expect a mean variance of around $s^2 = 0{\cdot}00460$. Then, taking the figures in Table 1, and defining the deviation,

$$\Delta = \log_{10}\text{observed} - \log_{10}\text{expected},$$

* An analysis was also made of the eighteen daily inter-replicate variances for both species. Applying Bartlett's test for the homogeneity of variances, the corrected χ^2 was for *P. aurelia*, 15·70 and for *P. caudatum*, 21·08, there being 17 degrees of freedom in each case. Assuming that this test is applicable to the present data, these would be regarded as perfectly satisfactory values of χ^2, and we would conclude that there was no evidence of any heterogeneity in the daily variances. This test, however, does not take into account the serial order of the observations, and there was a suggestion that the larger estimates of s^2 tended to occur in the earlier part of both series. When the individual variances were ranked in order of magnitude from large to small, and contrasted with the time order, 1, 2, 3, ..., 18, in which they were observed, the rank correlation (tau) was for *P. aurelia*, $\tau = +0{\cdot}42$, and for *P. caudatum*, $\tau = +0{\cdot}61$. Both of these rank correlations are significant ($P = 0{\cdot}02$–$0{\cdot}01$, and $P < 0{\cdot}001$, respectively). Thus, there is strong evidence that working in terms of $\log_{10} n_t$, the variance between replicates is some function of time. It might be held, therefore, that the use of the pooled inter-replicate variance, as a measure of the 'goodness-of-fit' between expected and observed, could be misleading. However, as we shall only compare this pooled variance with the mean square deviation between expected and observed over the same time period, this objection is perhaps not very serious. Any agreement between the two figures would suggest that, if there were any bias in the fitting, the degree of this bias was not very great.

we have the following mean square deviations, $\Sigma\Delta^2/18$, contrasted with those expected from the inter-replicate variances.

	$\Sigma\Delta^2/18$	Expected s^2
P. aurelia	0·00246	0·00194
P. caudatum	0·02738	0·00460

Table 1. *The observed and expected mean number of individuals* ($\bar{n}_i$) *in* 0·5 *c.c. of culture for populations of* Paramecium aurelia *and* P. caudatum *living alone*

Time (days)	*P. aurelia*		*P. caudatum*	
	Observed	Expected	Observed	Expected
2	14	12·2	10	9·4
3	34	26·6	10	17·7
4	56	56·0	11	32·1
5	94	110·9	21	54·5
6	189	197·1	56	84·1
7	266	301·6	104	115·8
8	330	395·1	137	143·0
9	416	458·5	165	162·2
10	507	493·9	194	174·0
11	580	511·5	217	180·6
12	610	519·7	199	184·1
13	513	523·5	201	186·0
14	593	525·2	182	186·9
15	557	526·0	192	187·4
16	560	526·3	179	187·6
17	522	526·4	190	187·7
18	565	526·5	206	187·8
19	517	526·6	209	187·8

Clearly, in the case of *P. aurelia*, the mean square deviation is of much the same order as that expected; while for *P. caudatum* it is markedly excessive, being very nearly six times greater than s^2. By inspection, it is clear from Table 1 that the major portion of this discrepancy is due to a marked divergence between 'expected' and 'observed' for days 4 and 5. For day 4 the expected numbers are nearly three times greater than those observed, and for day 5, 2·6 times. Obviously, as may be seen by graphing the figures in Table 1, no simple function such as the logistic could be expected to fit the recorded mean numbers of *caudatum* over the entire development of these replicates. However, apart from the entries for days 4 and 5, it appears that the fit for the remaining entries is reasonable, for we have for the remaining 16 days $\Sigma\Delta^2/16 = 0{\cdot}00658$, and the expected $s^2 = 0{\cdot}00404$.

Thus, excluding the results for days 4 and 5 in the case of *P. caudatum*, and judging by the criterion adopted, the overall discrepancy between the calculated and observed curves is in both cases much what one might expect from the degree of variation between replicates. There is, however, one further point which should be noted. If we examine the signs of the differences, 'observed' – 'expected', in Table 1, there is a suggestion that in both cases negative signs tend to occur more frequently in the earlier part of the growth curves, and positive signs more frequently in the later stages. In other words, these logistic curves may tend to overestimate the population in the earlier stages of the growth in numbers, and underestimate it in the later stages. The degree of this bias is, however, slight, and considering that the logistic is being applied merely as a first approximation, these curves are probably as good a fit to the observed data as might be expected from the use of such a simple model.

5. Growth in mixed cultures

When both species of *Paramecium* are living together in the same microcosm, we consider the pair of differential equations

$$\left.\begin{aligned}\frac{dN_1}{dt} &= (r_1 - a_1 N_1 - b_1 N_2)\, N_1, \\ \frac{dN_2}{dt} &= (r_2 - a_2 N_2 - b_2 N_1)\, N_2,\end{aligned}\right\} \tag{5·1}$$

in which r_1 and a_1 are the logistic parameters for the species S_1 when it is living alone, and similarly r_2 and a_2 those for the species S_2. The positive parameters b_1 and b_2 are a measure of the effect which each species has on the relative growth rate of the other.

Now, in the very simplified experimental conditions in which these two closely related types of *Paramecium* were living, we might assume that the magnitude of the effect which *P. aurelia* had on the relative rate of increase of *P. caudatum* was much the same as that which it had on its own relative rate of increase. And similarly for the effect of *P. caudatum* on *P. aurelia* in the mixed cultures. In other words, we might assume that we are dealing with a special case of (5·1) in which we have,

$$b_1 = a_2, \quad b_2 = a_1.$$

Then, by subtraction,

$$\frac{dN_1}{N_1 dt} - \frac{dN_2}{N_2 dt} = r_1 - r_2,$$

and integrating,*

$$\frac{N_1(t)}{N_2(t)} = \frac{N_1(0)}{N_2(0)}\, e^{(r_1 - r_2)t}. \tag{5·2}$$

Thus, if our assumptions are correct, two consequences should follow.

(1) If we define S_1 as the species having the greater intrinsic rate of increase ($r_1 > r_2$), then as $t \to \infty$, the numbers N_2 of the species $S_2 \to 0$, and ultimately the species S_1 will persist alone.

(2) If we take the natural logarithms of the ratios $N_1(t)/N_2(t)$ and plot these against time as the independent variable, then the successive points should fall on a straight line, and the slope of this line should be the same as the difference between the intrinsic rates of increase of the two species when each was living alone.

* For footnote see opposite page.

In the previous section we have estimated that the intrinsic rate of increase ($r = \log_e \lambda$) of *P. aurelia*, when living alone, was $r_1 = 0{\cdot}9125$; while for *P. caudatum*, $r_2 = 0{\cdot}7904$; the difference between these estimates being $r_1 - r_2 = 0{\cdot}1221$. We therefore define *P. aurelia* as the species of S_1 and *P. caudatum* as the species S_2, and we should expect that the latter would tend gradually to disappear from the mixed cultures. It is apparent from the data given by Gause (Table 3, p. 144) that this was the case in these experiments. In order to see whether the second consequence of our assumption is fulfilled, we must estimate the difference $(r_1 - r_2)$ from the data for the mixed cultures. Since the same proportion of the total population of each species was removed and discarded at each sample census, we may work in terms of the mean number of individuals ($\overline{n}_t$) per 0·5 c.c. of culture given by Gause.

Then defining

$$y_t = \log_e \overline{n}_1(t) - \log_e \overline{n}_2(t),$$

it will be found that the relation between the successive y_t and t is approximately linear, and fitting a straight line

$$y = kt \quad (k = r_1 - r_2)$$

passing through the origin (since $\overline{n}_1(0)/\overline{n}_2(0) = 1$ is given), the regression coefficient k was estimated from the 24 points available from day 2 to day 25 as

$$k = 0{\cdot}1132 \pm 0{\cdot}0034.$$

This value is remarkably close to the difference $r_1 - r_2 = 0{\cdot}1221$ which was estimated from the data for each species when living alone.

* This special case of the differential equations (5·1), when $b_1 = a_2$ and $b_2 = a_1$, was discussed by Volterra (1926) in his original memoir. If we substitute (5·2) for the terms in N_2 and N_1 in the first and second members, respectively, of (5·1), then the resulting equations can be integrated, and N_1 and N_2 can be expressed as functions of time. Thus, if we put

$$\left.\begin{aligned} N_1^{-1} &= x, \quad K_1 = r_1/a_1, \\ N_2^{-1} &= y, \quad K_2 = r_2/a_2, \end{aligned}\right\} k = r_1 - r_2,$$

these integrals can be written as,

$$x(t) = K_1^{-1} + K_2^{-1} C^{-1} e^{-kt} + C_1' e^{-r_1 t},$$
$$y(t) = K_2^{-1} + K_1^{-1} C e^{kt} + C_2' e^{-r_2 t},$$

where C, C_1' and C_2' are constants defining the initial state of the system.

These equations, however, are of little use in the present problem, since we also have to allow for the activities of the observer. In order to apply them to Gause's data, we must express them in the form

$$N_a(t+1) = f_a(N_1(t), N_2(t)) \quad (a = 1, 2).$$

This can be done by substituting from $x(t)$ in $x(t+1)$, and from $y(t)$ in $y(t+1)$, and using (5·2). If we write for simplicity,

$$\lambda_1 = e^{r_1}, \qquad \lambda_2 = e^{r_2},$$
$$\alpha_1 = (\lambda_1 - 1)/K_1, \quad \alpha_2 = (\lambda_2 - 1)/K_2,$$

we have finally, after a little rearrangement,

$$N_1(t+1) = \frac{\lambda_1 N_1(t)}{1 + \alpha_1 N_1(t) + \alpha_2 N_2(t)},$$

$$N_2(t+1) = \frac{\lambda_2 N_2(t)}{1 + \alpha_2 N_2(t) + \alpha_1 N_1(t)},$$

which is the system of equations used later in § 6.

The observed ratios of the mean number of individuals counted in the samples from the mixed cultures are given in Table 2, together with those expected from the equation

$$\frac{\overline{n}_1(t)}{\overline{n}_2(t)} = e^{0{\cdot}1132t}.$$

In order to see whether the departure from expectation was at all marked, the same procedure was adopted as in the previous section, and the pooled inter-replicate variance in

Table 2. *The observed and expected values of the ratio of the mean numbers,* Paramecium aurelia/P. caudatum, *growing in mixed culture*

Ratio $\overline{n}_1(t)/\overline{n}_2(t)$

Time (days)	Observed	Expected	Time (days)	Observed	Expected
2	1·00	1·25	14	5·78	4·88
3	1·91	1·41	15	6·44	5·46
4	2·00	1·57	16	6·60	6·12
5	1·84	1·76	17	8·07	6·85
6	2·30	1·97	18	7·46	7·67
7	1·60	2·21	19	6·55	8·59
8	1·78	2·47	20	7·00	9·62
9	3·15	2·77	21	8·25	10·77
10	2·95	3·10	22	17·50	12·07
11	4·59	3·47	23	17·50	13·51
12	3·64	3·89	24	9·43	15·13
13	6·18	4·36	25	17·50	16·94

Expected = $e^{0{\cdot}1132t}$.

terms of $\log_{10} n_t$ was estimated from the data for the mixed cultures. These variances were

P. aurelia 0·01319, *P. caudatum* 0·02423,

while the covariance was 0·0007267, each of these estimates being based on 36 degrees of freedom. Clearly, the correlation ($r = +0{\cdot}041$) between the numbers of the two species, when the time trend is eliminated, is negligible. Thus, since each $\overline{n}_t$ is in both cases based on three replicates, we should expect the variance of

$$y_t = \log_e \overline{n}_1(t) - \log_e \overline{n}_2(t),$$

to be given by

$$\text{var}\,(y_t) = \tfrac{1}{3}(2{\cdot}3026)^2\,(0{\cdot}01319 + 0{\cdot}02423) = 0{\cdot}06614.$$

From the fitting of the regression line passing through the origin, the discrepancy between observed (y) and expected (Y) was, for the 24 points,

$$s^2 = \Sigma(y - Y)^2/(n-1) = 0{\cdot}06213,$$

which is in close agreement with that expected from the variance observed between the replicates.

6. A FURTHER TEST OF THE HYPOTHESIS

There is another way of testing whether this simplified model of the competition equations is a satisfactory description of these data. In § 4 we estimated the parameters λ and α in the logistic equation,

$$N_{t+1} = \frac{\lambda N'_t}{1+\alpha N'_t},$$

from the data for each species when living alone. When, in the special case of (5·1), we have $b_1 = a_2$, $b_2 = a_1$, there is an analogous set of equations when the two species S_1 and S_2 are in a state of competition, viz.

$$\left.\begin{aligned} N_1(t+1) &= \frac{\lambda_1 N'_1(t)}{1+\alpha_1 N'_1(t)+\alpha_2 N'_2(t)}, \\ N_2(t+1) &= \frac{\lambda_2 N'_2(t)}{1+\alpha_2 N'_2(t)+\alpha_1 N'_1(t)}, \end{aligned}\right\} \qquad (6\cdot1)$$

where λ_1 and α_1 are the logistic parameters for the species S_1 when living alone, and similarly λ_2 and α_2 those for the species S_2. In both equations we have

$$N'_1(t) = 0\cdot9N_1(t) \quad \text{and} \quad N'_2(t) = 0\cdot9N_2(t),$$

in order to allow for the removal of one-tenth of the existing population whenever a sample census was taken. Then, given the numerical values estimated in § 4 for

$$\textit{P. aurelia}, \quad \lambda_1 = 2\cdot4905, \quad \alpha_1 = 0\cdot00026195,$$

$$\textit{P. caudatum}, \quad \lambda_2 = 2\cdot2042, \quad \alpha_2 = 0\cdot00058189,$$

we can calculate, given that $N'_1(0) = N'_2(0) = 20$, the average numbers of individuals in the mixed cultures at successive intervals of time, and therefore the expected mean number of each species in the samples which were removed and discarded from day 2 onwards.* These expected mean numbers can then be compared with the mean numbers actually observed by Gause in the mixed cultures. This is quite a stringent test of the model and of the simplifying assumption, since any error in the estimates of the parameters made from the data for each species when living alone would have a cumulative effect in any such step-by-step calculation, and the estimates of the expected mean numbers would tend to diverge more and more from those actually observed.

The results of this calculation are given in Table 3 from day 2 until day 25 in the experiments. It will be seen from Table 3 that in the case of *P. aurelia* the expected numbers follow closely those observed until about the sixteenth day, but that, certainly after day 19, they then tend to diverge from the latter. On the other hand, the expected curve for *P. caudatum* appears to give a reasonably good fit to the observed data throughout the entire period. The discrepancy between the observed and expected numbers is of much the same order of magnitude as that which might be expected from the inter-replicate variances.

* In carrying out this step-by-step calculation, it must be remembered that since no sample was taken at $t = 1$, we have $N'_1(1) = N_1(1)$ and $N'_2(1) = N_2(1)$.

Working in logarithms to the base 10 as before, and defining Δ as the deviation between log 'observed' and log 'expected' numbers in Table 3, we have

Species	Period (days)	$\Sigma\Delta^2/n$	s^2 expected from inter-replicate variance
P. aurelia	2–19	0·00511	0·00440
	2–25	0·00724	
P. caudatum	2–19	0·00775	0·00808
	2–25	0·00995	

Table 3. *The observed and expected mean number* ($\overline{n}_t$) *of individuals in* 0·5 *c.c. of culture in the mixed populations of* Paramecium aurelia *and* P. caudatum

Time (days)	*P. aurelia*		*P. caudatum*	
	Observed	Expected	Observed	Expected
2	10	11·7	10	9·2
3	21	24·5	11	17·0
4	58	47·9	29	29·4
5	92	84·7	50	46·0
6	202	131·8	88	63·3
7	163	179·9	102	76·5
8	221	221·0	124	83·2
9	293	253·2	93	84·4
10	236	278·4	80	82·1
11	303	299·1	66	78·1
12	302	317·1	83	73·3
13	340	333·5	55	68·2
14	387	348·8	67	63·1
15	335	363·2	52	58·1
16	363	376·8	55	53·3
17	323	389·7	40	48·8
18	358	401·7	48	44·5
19	308	413·0	47	40·5
20	350	423·5	50	36·8
21	330	433·2	40	33·3
22	350	442·2	20	30·1
23	350	450·5	20	27·1
24	330	458·1	35	24·4
25	350	465·1	20	21·9

It is quite evident that up to day 19, in the case of both species, the discrepancy between the figures in Table 3 is no greater than that which might be expected from the degree of variation between the three replicates on which the means are based. Even up to day 25 the average overall agreement is reasonable; but, as has been mentioned earlier, it is evident by inspection that in the case of *P. aurelia*, from about day 16, the expected and observed curves are diverging. However, considering the approximate nature of the parameters estimated from the data for each species when living alone, it is remarkable how closely we have been able to anticipate, as it were, the results actually observed in the experiments when both species were living together in the same microcosm.

7. CONCLUSION AND SUMMARY

It seems reasonable to conclude, as a result of this analysis, that the deterministic model associated with the names of Lotka and Volterra is a remarkably good description of the changes which occurred in the mean values of these stochastic processes. The two main points leading to this conclusion are

(1) The first approximation in this model for a species living alone—namely the logistic equation—is a good fit to the data for *P. aurelia*, as judged by a comparison of the mean square deviation between expected and observed with the variance between replicates. In the case of *P. caudatum* this equation was not a good fit; but this was due almost entirely to a relatively large discrepancy between expected and observed on 2 days. If these were neglected, the fit of the logistic to the remaining observations for this species would be reckoned as satisfactory.

(2) Given the logistic parameters for the two species when living alone, and assuming that a special case of the competition equations in this model were applicable to these closely related types of Protozoa, it should theoretically be possible to calculate the expected numbers in the sample censuses when the two species were competing together for a common food supply. The agreement between the expected numbers calculated in this way and those actually observed was very much what might be expected from the variation between replicates in the competition experiments.

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ON THE DISTRIBUTION OF *TRIBOLIUM CONFUSUM* IN A CONTAINER*

By D. R. COX and W. L. SMITH

1. Introduction

Neyman, Park & Scott (1956) have recently discussed a number of statistical problems arising from studies of the flour beetle, *Tribolium*. One experiment they describe was as follows. A container was filled with fresh flour occupying a volume of a cube $10 \times 10 \times 10$ in. On the surface of the flour a large number of *Tribolium confusum* beetles was placed. The container was kept for about four months in a dark incubator maintaining an approximately constant temperature and humidity, and was periodically rotated. After the 4 months the contents of the container were divided into 1000 cubes $1 \times 1 \times 1$ in. and the contents of each small cube were lifted separately and examined.

Table 1. *Distribution of* Tribolium *over a plane cross-section (averaged experimental data)*

First quarter	Second quarter				
	3·2 2·5	3·5 1·8	4·0 2·0	6·4 3·7	12·8 14·3
	3·5 1·8	5·0 1·3	4·2 2·2	6·8 3·3	11·8 12·3
Third quarter	4·0 2·0	4·2 2·2	5·2 2·1	8·0 4·0	11·6 12·5
	6·4 3·7	6·8 3·3	8·0 4·0	9·2 7·2	13·8 18·6
	12·8 14·3	11·8 12·3	11·6 12·5	13·8 18·6	17·9 30·4

Obtained from the data quoted by Neyman *et al.* (1956) by the averaging process described in the text. The upper figure is the density of females, the lower figure the density of males. The average density is symmetrical in the square and only the distribution in the fourth quarter is shown (note that the averaged distribution is symmetrical also about the diagonal of the whole square).

The general feature of the experimental results is, on any given 'layer', a gradual increase in the density of the beetles towards the sides of the container, with the density highest in the corners. Table 1 is a contraction of Fig. 10 of Neyman *et al.* (1956) obtained by arranging their data on the assumption that, apart from random fluctuations, the density of beetles has the full symmetry properties of the square. Thus, the value given in the right-hand corner of Table 1 is the average of the four 'corner' values given by Neyman *et al.* (1956).

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The distributions for males and females are somewhat different in that the density for females falls steadily towards the centre of the square from a relatively low maximum at the corners, while the density for the males falls more sharply from a high maximum at the corner and is substantially constant in the central portion. The general feature of a gradual increase in density towards the edges and along the edges towards the corners is, however, the same for both sexes.

After discussing some possible models to explain these phenomena, Neyman *et al.* (1956) concluded that, up to the time of their writing, no model producing the gradual increase in density was available. The object of the present note is to discuss a simple probability model that does produce a density distribution which is qualitatively similar to the one observed.

2. Previous work

Broadbent & Kendall (1953) had some success in describing the two-dimensional motion of *Trichostrongylus retortaeformis* by simple Brownian motion. Sherman investigated the consequences of such a model in the present context, and showed that a random walk over a square lattice with inelastic boundaries could produce a concentration of beetles at the boundaries and in the corners, but could not account for the gradual increase in density towards the edges. In a further investigation (Sherman, 1956) of the one-dimensional random walk, he showed that distributions more like the observed one could be obtained if a suitable boundary condition was inserted. This was that after striking the boundary, the moving point remains there a randomly distributed time, and is then placed instantaneously a finite distance within the region of motion. Sherman did not, however, claim this boundary condition to be a reasonable explanation of the experimental results.

3. Possible models

One approach is to assume that the motion of the beetle is described by a more complicated random walk whose steps are not infinitesimal, and may indeed be correlated in direction and length; see, for example, the type of walk investigated by Daniels (1952). Suppose that this is combined with a boundary condition of the following type. On meeting the boundary the beetle remains there for a time-interval which has a certain frequency distribution, and it then pursues a path that is the reflexion in the boundary of the path that would have been followed in the absence of the boundary.

It may be shown by the method of images that for any such walk, without drift, which leads in the case of unrestricted motion to an asymptotic normal distribution of indefinitely increasing dispersion, the limiting distribution within the square boundary is uniform except for a line concentration on the boundary. This result applies also for such a two-dimensional motion within a circular barrier, and is probably true for a very wide class of boundaries.

One objection to the Brownian motion models is that the path of the beetles is highly irregular, as though the beetle were constantly forgetting its direction. This objection does not apply to the generalizations just discussed; the failure of the generalizations to produce the required behaviour must be connected with the use of an inappropriate boundary condition. Indeed, the assumption of perfect reflexion seems a most unrealistic description of the true behaviour of *Tribolium*.

In the present note we shall show that by taking a more realistic boundary condition, even with a grossly oversimplified description of the beetles' paths within the flour, we can obtain a distribution of beetles of the type observed.

4. General description of the model

The basic assumption is that the motion is two-dimensional and that within the flour a beetle follows a straight path. When it meets the boundary it may (i) with probability p, return along its original path; or, (ii) with probability $1-p$, move a distance τ, possibly zero, along the boundary before choosing a new direction of motion, independently of its previous direction.

The distance τ and the angle θ that the new direction of motion makes to the normal to the boundary at the point of departure are assumed to have distribution functions $H(\tau)$ and $G(\theta)$. We assume for simplicity that the beetle moves with constant speed, although this assumption may be greatly generalized without affecting the final results.

In spite of the simplicity of the model which we have just described, we have been unable to obtain an explicit solution to the problem of limiting beetle distribution within a square boundary. We shall, therefore, discuss first the solution for a circular container.

5. Solution for a circular container

Assume that the motion takes place within a circle of unit radius. Let γ_r denote a concentric circle of radius $r < 1$. Let Ω denote a path, defined to be the total track of the beetle between two successive selections of a new direction of motion within the flour, i.e. a path consists of one chord of the circle, possibly traversed several times, followed by an arc of the circumference of the unit circle.

Consider a large number of such paths developed according to the distributions $H(\tau)$, $G(\theta)$ defined above. The probability that a beetle is inside γ_r is then the total length of all parts of paths Ω within γ_r, divided by the total length of all paths. Thus, it is easy to see that the probability $\Pi(r)$, say that a particular beetle is within γ_r is given by

$$\Pi(r) = \frac{\text{mean length of that part of a path which is in } \gamma_r}{\text{mean length of a path}} = \frac{\rho(r)}{\lambda}, \quad \text{say.} \tag{1}$$

This argument assumes that the spatial distribution of the beetles comes to statistical equilibrium and that probabilities and time-averages can be equated. These points can be justified, and (1) derived directly, by the theory of regenerative stochastic processes (Smith, 1955).

The problem therefore resolves itself into the geometrical question of the value of $\rho(r)$. To calculate $\rho(r)$ note from Fig. 1 that the chord length in γ_r is $2\sqrt{(r^2-\sin^2\theta)}$ and that, after making allowance for the probability of traversing the chord several times,

$$\rho(r) = \frac{2}{1-p}\int_0^{\sin^{-1}r} (r^2-\sin^2\theta)^{\frac{1}{2}}\, dG(\theta). \tag{2}$$

Similarly,

$$\lambda = \bar{\tau} + \frac{2}{1-p}\int_0^{1} (r^2-\sin^2\theta)^{\frac{1}{2}}\, dG(\theta), \tag{3}$$

where $\bar{\tau}$ is the mean of the distribution $H(\tau)$.

To evaluate these integrals, we shall write

$$G'(\theta) = a + b\,|\sin\theta| + c\sin^2\theta; \tag{4}$$

where higher terms could be calculated, if required. Write

$$K(r) = \int_0^{\frac{1}{2}\pi} (1 - r^2 \sin^2\theta)^{-\frac{1}{2}}\, d\theta, \tag{5}$$

$$E(r) = \int_0^{\frac{1}{2}\pi} (1 - r^2 \sin^2\theta)^{\frac{1}{2}}\, d\theta, \tag{6}$$

for the complete elliptic integrals of the first and second kinds, respectively. Also put

$$O_k(r) = \int_0^r \frac{t^k\, dt}{\sqrt{[(1-t^2)(1-r^2t^2)]}}. \tag{7}$$

It is then easily shown from (4) that

$$\rho(r) = \frac{2r^2}{1-\rho}\{a[O_0(r) - O_2(r)] + br[O_1(r) - O_3(r)] + cr^2[O_2(r) - O_4(r)]\}. \tag{8}$$

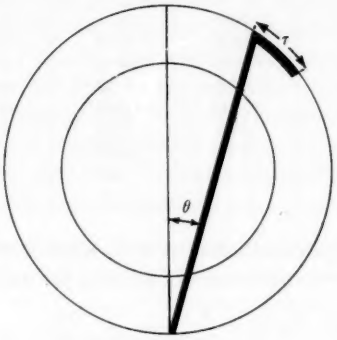

Fig. 1. A typical path.

The functions $O_k(r)$ satisfy simple recurrence relations (Byrd & Friedman, 1954) and for $k = 0$ through 4 they are given by

$$O_0(r) = K(r), \qquad O_3(r) = \frac{1}{2r^3}\left[\frac{1+r^2}{2}\log\frac{1+r}{1-r} - r\right],$$

$$O_1(r) = \frac{1}{2r}\log\frac{1+r}{1-r}, \qquad O_4(r) = \frac{1}{3r^4}[(2+r^2)K(r) - 2(1+r^2)E(r)].$$

$$O_2(r) = \frac{1}{r^2}[K(r) - E(r)],$$

From these results we can find $\Pi(r)$ in terms of tabulated functions. A more useful expression is for the density of beetles per unit area as a function of r. This is given for $r < 1$ by

$$\tilde{\omega}(r) = \frac{1}{2\pi r}\Pi'(r). \tag{9}$$

We obtain

$$\tilde{\omega}(r) = \frac{aK(r) + \frac{1}{2}b\log\frac{1+r}{1-r} + c[K(r) - E(r)]}{\pi[2a + b + \frac{2}{3}c + \bar{\tau}(1-p)]}. \tag{10}$$

The simplest, and physically most plausible distribution in the family (4) has $b = c = 0$, $a = 2/\pi$, i.e. is the rectangular distribution for θ. For this distribution

$$\bar{\omega}(r) = \frac{K(r)}{2\pi\left[1 + \frac{\pi\bar{r}(1-p)}{4}\right]} \quad (r < 1). \tag{11}$$

A short table of $K(r)$ is given in Table 2 and shows that the density per unit area increases steadily from the centre. In the third and fourth columns of Table 2 are given the functions which determine the density in the cases $a = c = 0$ and $a = b = 0$, i.e. when the density function of θ is proportional to $|\sin\theta|$ and to $\sin^2\theta$.

Table 2. *Density per unit area (unstandardized) for some special cases*

r	$K(r)$	$\log\frac{1+r}{1-r}$	$K(r) - E(r)$
0·0	1·57	0	0
0·2	1·59	0·41	0·03
0·4	1·61	0·85	0·13
0·6	1·75	1·39	0·33
0·8	2·00	2·20	0·72
0·9	2·28	2·94	1·11
0·95	2·59	3·66	1·49
1·00	∞	∞	∞

In addition to the continuous distribution over the interior of the unit circle, there is a line density of probability on the circumference of amount per unit length

$$\tfrac{1}{2}[1 - \Pi(1)] = \frac{1}{2\pi}\frac{\bar{r}(1-p)}{2a + b + \frac{2}{3}c + \bar{r}(1-p)}. \tag{12}$$

The general conclusions to be drawn from these results are as follows. If the distribution of θ is uniform, i.e. if the beetles select paths 'randomly' oriented, then the resulting density is nearly constant for $r < \frac{1}{2}$ and increases steadily to infinity as r tends to 1. If the probability density of θ is proportional to $|\sin\theta|$, i.e. if there is a fairly strong tendency to select paths nearly tangential to the boundary, the resulting distribution of beetles is naturally much more strongly concentrated at the larger values of r, rising from a zero density at the centre. Some distributions of θ showing a slight tendency to concentrate on the tangential paths can be obtained by suitable values of a, b and c. The resulting distributions of beetles are then given by a weighted combination of the columns of Table 2 and therefore have densities increasing with r. Certain distributions of θ attaching relatively greater probability to the paths normal to the boundary can be represented by negative values of b and c, provided only that (4) is always positive and integrates to unity. In these cases the form of $\bar{\omega}(r)$ depends, of course, on the relative magnitudes of the coefficients.

6. Solution for the square

As we have remarked above, the mathematical treatment for the square container is much more difficult than for the circular container. This is primarily because it is first of all necessary to find the distribution of starting points of paths along the edges of the square. This distribution is determined by an integral equation which we have been unable to solve.

Instead of an exact mathematical treatment of the case of a square container we have been obliged to introduce approximate methods. We assumed that paths may start and end only at a discrete set of points along the sides of the square. We have made the further assumptions that θ is uniformly distributed and that there is no motion along the boundary, $\bar{\tau} = 0$. By a tedious procedure it is possible to work out the equilibrium distribution of beetles over the square. A brief outline of the steps in these calculations follows.

(i) Eight boundary points were symmetrically placed at distances $\frac{1}{2}, \frac{3}{2}, \ldots, \frac{15}{2}$ from the ends of the sides of a square of side 8. These points were then numbered $1, 2, \ldots, 8$ cyclically.

(ii) Transition probabilities giving the probability that a path starting from a point j on one side would strike one of the other sides within a distance $\frac{1}{2}$ of a point k were calculated on the basis of a uniform distribution for θ. It was then assumed that any path, which ended within a distance $\frac{1}{2}$ of a point k, actually ended at k.

(iii) Symmetry considerations show that the final distributions can be obtained by working with paths starting from points 1, 2, 3, 4 only, on one side. Therefore, a Markov chain with four states was involved and a system of linear equations determined the equilibrium probability that a path starts from a position with a specified number.

(iv) The probability attached to each possible path was then calculated using (iii) and (ii).

(v) The interior of the square was divided into suitable areas and the length of each path within each area tabulated. When multiplied by the correct probability from (iv) these lengths gave the numerator in the expression analogous to (1). The mean length of all paths was determined in a similar way, and hence the equilibrium probabilities of finding a beetle in the various areas of the square were determined.

Table 3. *Equilibrium density of beetles over a square, for a simple model*

First quarter	Second quarter			
Third quarter	0·0131 0·0125 0·0135 0·0176	0·0125 0·0139 0·0144 0·0167	0·0135 0·0144 0·0144 0·0183	0·0176 0·0167 0·0183 0·0227

The probabilities found in this way are shown in Table 3. The distribution is only given for the fourth quarter of the square because of the symmetry involved.

Notice that the general features of Table 3 are the same as those of the experimental results in Table 1. That is, the density is fairly constant in the centre of the square and rises steadily towards the sides. Along each edge the density rises to maxima in the corners.

However, there are two discrepant features. First, there are puzzling minor irregularities in the computed values in Table 3. For example, the density in the centre of the square is slightly higher than that below or to the side of the centre. There are three possible explanations of these irregularities.

(i) They may represent real features of the random motion, in the sense that they would be present in the solution of the correct 'continuous' model where the paths are not restricted to a discrete set.

(ii) They may be the result of numerical errors that have escaped the various checking schemes used.

(iii) They may be a consequence of the 'discrete' approximation adopted. Thus, we have replaced the whole set of paths starting from points in an interval on one side of the square, and going to points in an interval on another side of the square, by one path. This single path will, in general, contribute zero probability to regions that, in the continuous model, would receive appreciable probability. While this unbalance should tend to even itself out where there are many paths, it is reasonable to expect some irregularities to remain.

We are of the opinion that (i) and (ii) are unlikely, and that (iii) is the correct explanation.

The second discrepancy is that the magnitudes of the trends in Table 3 do not agree too well with the experimental values. The corner density in Table 3 is less than twice the density in the centre, whereas the corresponding ratios for the experimental densities are about 4 for females and 15 for males. The theoretical trend along the edge from the centre is about correct for females and somewhat too small for males. Some of this lack of agreement could be accounted for by the kind of assumption, made in § 5, that a beetle which has reached the boundary tends to remain there for some time before choosing a new path within the flour.

7. A more realistic model

The major artificiality in the model just analysed is the assumption that the paths are straight in the interior of the container. A more realistic model would be the following. At the beginning of a new path from the boundary the beetle chooses a starting angle θ as in the earlier model. However, let it now be assumed that in an element δx of the path there is a constant chance $\mu\delta x + o(\delta x)$ that a change of direction will occur. Also, suppose that if a change does occur then it is uniformly distributed over $(0, 2\pi)$. The beetle then moves along in the new direction until either it meets the boundary or a further change in direction occurs. Such a model would have three adjustable features.

(i) The parameter μ which determines the average length of straight sections of path.

(ii) The distribution of boundary starting angles θ.

(iii) The magnitude of any tendency to stay near the boundary for some time before choosing a new path.

It should be possible to obtain a mathematical expression for the way the equilibrium density of beetles depends on these features. However, we have been unable to obtain such an expression. Certain qualitative properties of the model are more or less obvious. If μ is small we have nearly the case which we have discussed in the main part of the present paper. If μ is large we have very nearly Brownian motion, leading to a uniform distribution, except for a line concentration in the boundary. Moreover, it is fairly clear what the general effect of changes from a uniform distribution of θ would be. If values of θ near $\frac{1}{2}\pi$ were relatively frequent the density near the edges of the square is increased relative to that at the centre. A tendency to remain stationary near the boundary would also, obviously, increase the density in the regions adjacent to the boundary.

It seems plausible that a good fit to both sets of data in Table 1 could be obtained by suitably adjusting the three features listed above, and moreover that a reasonable fit is possible even with the simple uniform distribution of θ. This would explain the experimental results without appeal to a special hypothesis that the beetles' motions are different near to and distant from the surface.

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THE χ^2 GOODNESS-OF-FIT TEST FOR NORMAL DISTRIBUTIONS

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SUMMARY. Chernoff & Lehmann (1954) have shown that if, in making a χ^2 goodness-of-fit test of a continuous distribution, the parameters are estimated efficiently from the sample and not from the cell frequencies, the statistic X^{2*} is asymptotically distributed as

$$\chi^2_{k-s-1}+\lambda_1 y_1^2+\ldots+\lambda_s y_s^2, \quad \text{where} \quad 0\leqslant\lambda_1, \ldots, \lambda_s\leqslant 1, \; y_1, \ldots, y_s$$

being independent standard normal variables. They used k fixed class intervals, defined without reference to the sample. In the present paper, the particular case of the normal distribution is examined when the class intervals are chosen to contain, with reference to the sample mean and variance, constant probabilities and so vary with the sampling. The X^2 statistic is again found to be distributed in the Chernoff & Lehmann form ($s = 2$). Explicit formulae are given for λ_1 and λ_2 which tend to zero rapidly as k increases. It is suggested that at least ten class intervals be used in practice so that the tabular points of χ^2 may be used with an error of less than 1 %.

1. Introduction

Fisher (1924) first established the fundamental theorem of χ^2 goodness-of-fit tests—that the X^2 statistic is asymptotically distributed as χ^2, the degrees of freedom being the number (k) of classes less one less the number (s) of parameters estimated. Cramér's well-known text (1946) gives the most rigorous proof available but, like Fisher's argument, it is formulated for a multinomial situation, i.e. there are k classes with probabilities $p_l(\theta_1, \theta_2, \ldots)$, $l = 1, \ldots, k$ and the observations consist of class frequencies $n_1, n_2, \ldots, n_k$, $\Sigma n_l = N$. Provided the parameters admit regular efficient estimators (which are functions of $n_1, n_2, \ldots, n_k$) the result follows.

In fitting a continuous distribution with density $f(x; \theta_1, \theta_2, \ldots)$, k class intervals $(-\infty, Z_1)$, (Z_1, Z_2), $\ldots$, (Z_{k-1}, ∞) could be used. Then

$$p_l(\theta_1, \theta_2, \ldots) = \int_{Z_{l-1}}^{Z_l} f(x; \theta_1, \theta_2, \ldots)\, dx. \tag{1}$$

Suppose $n_1, n_2, \ldots, n_k$ are the observed frequencies in these intervals. Then, if $Z_1, Z_2, \ldots, Z_{k-1}$ are fixed, and if the class frequencies rather than the sample values $x_1, x_2, \ldots, x_N$ are used efficiently to estimate $\theta_1, \theta_2, \ldots$, Fisher's result may be used, provided Cramér's conditions are satisfied.

It is, however, probably universal practice in this situation to use the sample values $x_1, x_2, \ldots, x_N$ to provide efficient estimates $\hat{\theta}_1, \hat{\theta}_2, \ldots$ of $\theta_1, \theta_2, \ldots$.

These are not the estimators envisaged above which are functions of the cell frequencies. They contain more information and thus are more efficient. Fisher, in his 1924 paper, showed that if, in the multinomial problem, estimators were used with efficiencies differing from the efficiencies of the maximum likelihood (or minimum χ^2) estimators based on frequencies, the statistic would not have the χ^2 distribution. Neyman & Pearson (1928, 1931), and Sheppard (1929) examined the situation again and showed that X^2 is larger than χ^2_{k-s-1}. The detailed solution to the problem was first given by Chernoff & Lehmann (1954). For the problem of testing the goodness-of-fit of a continuous distribution, they assumed

* Following Cochran (1952), we use X^2 to denote the statistic $\Sigma(f_o - f_e)^2/f_e$ and chi-square (or χ^2) for the well-known distribution (or a variable with this distribution).

that the cell boundaries $Z_1, Z_2, \ldots, Z_{k-1}$ are fixed and $\theta_1, \theta_2, \ldots, \theta_s$ were estimated efficiently from the sample $x_1, x_2, \ldots, x_N$ and their result showed that X^2 was distributed as

$$\chi^2_{k-s-1} + \lambda_1 y_1^2 + \ldots + \lambda_s y_s^2,$$

where $\lambda_1, \ldots, \lambda_s$ are certain latent roots, all in the interval $(0, 1)$ and $y_1, \ldots, y_s$ are standard normal variables, independent of each other and of χ^2_{k-s-1}. This shows explicitly why X^2 is greater than χ^2_{k-s-1}—the fact observed by the earlier writers. They applied their general theorem to the normal distribution and showed that the numerical effects may be important, e.g. that the significance level may well be twice the tabular value.

Since, with Chernoff & Lehmann's formulation, the distribution of X^2 depends on the unknown parameters, it is hard to see the practical significance of their result. If, however, the cell probabilities $p_1, p_2, \ldots, p_k$ are prescribed and the class intervals chosen so that, always,

$$p_l = \int_{Z_{l-1}}^{Z_l} f(x; \hat{\theta}_1, \hat{\theta}_2, \ldots)\, dx, \tag{2}$$

then it will be shown below for the case of the normal, that the distribution of X^2 does not depend on the values of θ_1, θ_2, but it is still not that of χ^2; in fact, it is a special case of that given by Chernoff & Lehmann. The relevance of this formulation—fixed cell probabilities rather than fixed cell boundaries—to practice is not immediately obvious. Before defending it, a survey of current practical methods is necessary.

Even after restricting discussion to the normal distribution, it is not easy to define the current practices for the choice of class intervals. The most usual method may be seen in the examples of Cramér (1946, §§ 30–34). The intervals, except for the two extreme ones, are of equal length. This has the advantage that the determination of the class frequencies is easy if the end-points are simply related to the scale of measurement, and also makes the histogram simple to assess visually. The number of class intervals is restricted above by the requirement that the expected class frequencies are not to be 'too small' (i.e. to avoid deficiencies in the asymptotic theory due to small sample effects) and below by consideration of loss of sensitivity (i.e. power). While there is no rule for positioning the class boundaries, it is clear that their position is related to the mean of the sample, and that the length of the class intervals is conditioned by the variance. Often this means only that the class intervals are determined by inspection of the data. Since the data varies from sample to sample, the class intervals may well do so. The restrictions on the expected class frequencies are restrictions on the probability content of the cells, although they do not require the probability contents of the cells to be fixed. With a large mass of data, possibly the simplest method of obtaining class intervals with 'reasonable' probability contents (i.e. 'reasonable' expected frequencies), is to compute the sample mean and variance, $\bar{x}$ and s^2, and construct intervals centred on $\bar{x}$ with lengths of some chosen multiple of s. This requires no judgement and the boundaries can be altered slightly, if necessary, to coincide nearly with the scale of measurement. If this is done, we have exactly, or approximately, the formulation (2). Another case where our formulation is exactly correct will be given below when the work of Mann & Wald (1942) is discussed. Thus our formulation is very closely related to practical methods.

To arrive at something which can be analysed mathematically and which is similar to this procedure, the following rule is proposed. For k even, let the sample mean be a class

boundary, and let an equal number of intervals be marked out on either side of it of a length equal to some multiple of the standard deviation of the sample. For k odd, let the sample mean be at the centre of a class interval and let an equal number of intervals be marked out on either side of it of length equal to the central interval and to some multiple of the sample standard deviation. A strict symmetry of the class intervals about the sample mean is used in these rules as it leads to mathematical simplicity. In practice, of course, this symmetry is usually only approximate as it conflicts with the desire to have the class boundaries simply related to the units of measurement.

Mann & Wald (1942) have argued that for reasons of power, the class intervals should have equal probability content. They have given a formula for the number of intervals k, in terms of N and the significance level to be used, again based on power. These rules are proved for an arbitrary distribution, *without* parameters requiring estimation. This restriction does not affect the argument for the equal probability intervals and this part of the rule has certainly (see § 5) been used for the present problem. In fact, it also has the merit of numerical simplicity since all the denominators in X^2 are equal. Clearly, the formulation is that of (2), with, in addition, $p_l = 1/k$ $(l = 1, 2, \ldots, k)$. It is easily analysed by the present approach since the class intervals are symmetrical about the sample mean. The formula for k is certainly no longer valid and, without making an investigation of this point, the only assertion that can be confidently made is that many more class intervals should be used than is now customary. The present paper provides a quite different reason for this recommendation, as will be seen later when the equal probability and equal length interval rules are examined.

As has been remarked by Chernoff & Lehmann and the earlier writers, the effect on the distribution of X^2 of different estimators is not entirely a matter of their efficiency. The replacement of a parameter by an estimator usually increases the variability of the results. If, in the present problem, an estimator found from the sample under test is used, there is a contrary tendency, since the hypothesis is forced to conform to the sample. To differentiate between these opposing effects, several cases (which do not seem to arise in practice) have been considered, where independent estimators of the mean and variance, also from samples of N, are available. It is found, for k large, that the mean and the variance behave in the same way—an independent estimator increases, roughly, the degrees of freedom by unity, an estimator from the sample under test decreases the degrees of freedom by unity, from the value when the parameter is known.

The discussion in this paper is limited to the asymptotic distribution obtained when a normal distribution is fitted. In further papers a general asymptotic theorem will be proved and applied to other distributions, some of which show serious, and others trivial, underestimation of the Type I Error in X^2 goodness-of-fit tests. Sampling investigations are planned to investigate the small sample phenomena.

2. Distribution of X^2—general case

In the notation of § 1, let $p_1, p_2, \ldots, p_k$ be the areas desired above the class intervals, as defined in (2) and let $n_1, n_2, \ldots, n_k$ $(\Sigma n_l = N)$ be the class frequencies; then the statistic whose distribution is required is

$$X^2 = \sum_{l=1}^{k} \frac{(n_l - Np_l)^2}{Np_l}. \tag{3}$$

For this purpose, let indicator variables be defined as follows:

$$Y_i(l) = \begin{cases} 1 & \text{if } x_i \text{ falls in } (Z_{l-1}, Z_l) \\ 0 & \text{if } x_i \text{ does not fall in } (Z_{l-1}, Z_l), \end{cases} \tag{4}$$

where $Z_0 = -\infty$, $Z_k = +\infty$. Then

$$n_l = \sum_{i=1}^{N} Y_i(l), \tag{5}$$

where the joint distribution of $Y_1(l), Y_2(l), \ldots, Y_N(l)$ is unchanged by permutation of the subscripts. Thus

$$E(n_l) = NP(Y_1(l) = 1), \tag{6}$$

$$E(n_l^2) = NP(Y_1(l) = 1) + N(N-1)\,P(Y_1(l)\,Y_2(l) = 1), \tag{7}$$

$$E(n_l n_m) = N(N-1)\,P(Y_1(l)\,Y_2(m) = 1). \tag{8}$$

Since, by (5), the n_l $(l = 1, \ldots, k)$ are expressed as sums of N variables, it will in general be true when N tends to infinity that the joint distribution of $n_1, \ldots, n_k$ is multivariate normal with a mean vector and covariance matrix obtainable from (6), (7), (8). X^2 may be written as

$$\begin{aligned} X^2 = \Sigma \frac{(n_l - E(n_l))^2}{Np_l} &+ 2\Sigma \frac{(n_l - E(n_l))\,(E(n_l) - Np_l)}{Np_l} \\ &+ \Sigma \frac{(E(n_l) - Np_l)^2}{Np_l}. \end{aligned} \tag{9}$$

The method will be to show from the joint distribution of the n_l's that the last two terms on the right-hand side of (9) contribute nothing to the limiting ($N = \infty$) distribution of X^2, while the first term is distributed as a quadratic form in normal variables. For this it is necessary to determine $P(Y_1(l) = 1)$ and $P(Y_1(l)\,Y_2(m) = 1)$ to terms of order $O(N^{-1})$.

This programme will now be carried out for the normal distribution. It is clearer to divide this into two sections—first when only the mean requires estimation, and second when both the mean and the variance require estimation.

3. Normal distribution, mean unknown, variance known

Let $z_1, z_2, \ldots, z_{k-1}$ be such that

$$p_l = \int_{z_{l-1}}^{z_l} \phi(t)\,dt, \tag{10}$$

where $z_0 = -\infty$, $z_k = +\infty$, $\phi(t) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}t^2)$. If the unknown mean is estimated by the mean $\bar{x}$ of the sample under test, it is clear that the Z_i of § 2 are simply $z_i + \bar{x}$. Hence, n_l is the number of $x_1, x_2, \ldots, x_N$ in $(z_{l-1} + \bar{x}, z_l + \bar{x})$, which is the same as the number of $x_1 - \bar{x}, \ldots, x_N - \bar{x}$ in (z_{l-1}, z_l). The indicators will be defined by

$$Y_i(l) = \begin{cases} 1 & \text{if } x_i - \bar{x} \text{ falls in } (z_{l-1}, z_l), \\ 0 & \text{if } x_i - \bar{x} \text{ does not fall in } (z_{l-1}, z_l). \end{cases} \tag{11}$$

It is convenient to use $t_i = (x_i - \bar{x})/\sqrt{(1 - 1/N)}$ since then $t_1, t_2, \ldots, t_N$ are jointly normal, with zero means, unit variances and correlations $\rho = -1/(N-1)$. Thus the joint density of t_1 and t_2, to the required order, is given by

$$\frac{\exp\{-\frac{1}{2}(t_1^2 + t_2^2)\}}{2\pi}\{1 + \rho t_1 t_2\}. \tag{12}$$

To make the subsequent formulae simpler, it is convenient to introduce the notation

$$\Phi_r(l, N) = \int_{z_{l-1}(1-1/N)^{-\frac{1}{2}}}^{z_l(1-1/N)^{-\frac{1}{2}}} t^r \phi(t)\, dt. \tag{13}$$

For brevity the arguments l and N are only shown when there is any doubt. From (11), $P(Y_1(l)=1)$ and $P(Y_1(l)\,Y_2(m)=1)$ are obtained by integrating (12). Thus

$$\left.\begin{aligned} P(Y_1(l)=1) &= \Phi_0(l), \\ P(Y_1(l)\,Y_2(m)=1) &= \Phi_0(l)\,\Phi_0(m)+\rho\Phi_1(l)\,\Phi_1(m). \end{aligned}\right\} \tag{14}$$

With the notation

$$\psi_r(l) = z_l^r\,\phi(z_l) - z_{l-1}^r\,\phi(z_{l-1}) \tag{15}$$

it is easily seen that, to the orders in which they will be required,

$$\left.\begin{aligned} \Phi_0 &= p_l + \frac{\psi_1}{2N}, \\ \Phi_1 &= -\psi_0 + \frac{\psi_2}{2N}, \\ \Phi_2 &= p_l - \psi_1. \end{aligned}\right\} \tag{16}$$

Hence

$$\left.\begin{aligned} E(n_l) &= Np_l + \psi_1(l)/2, \\ \operatorname{var}(n_l) &= N(p_l - p_l^2 - \psi_0^2(l)), \\ \operatorname{covar}(n_l, n_m) &= N(-p_l p_m - \psi_0(l)\,\psi_0(m)). \end{aligned}\right\} \tag{17}$$

Putting $\boldsymbol{\psi}_r' = [\psi_r(1), \ldots, \psi_r(k)]$, $\mathbf{1}' = [1, \ldots, 1]\,(1 \times k)$ and $\mathbf{D}(p_l) \equiv \mathbf{D}$ for the $k \times k$ matrix with the p_l down the main diagonal in order, the covariance matrix of the n_l's may be written as

$$\boldsymbol{\Lambda}_1 = N\{\mathbf{D} - \mathbf{D}\mathbf{1}\mathbf{1}'\mathbf{D} - \boldsymbol{\psi}_0\boldsymbol{\psi}_0'\}. \tag{18}$$

Thus, asymptotically, the second last term of (9) is a normal variable with zero mean and variance

$$\frac{1}{N}\boldsymbol{\psi}_1'\mathbf{D}^{-1}\{\mathbf{D} - \mathbf{D}\mathbf{1}\mathbf{1}'\mathbf{D} - \boldsymbol{\psi}_0\boldsymbol{\psi}_0'\}\mathbf{D}^{-1}\boldsymbol{\psi}_1,$$

and so converges stochastically to zero as $N \to 0$, while the last term is equal to $\boldsymbol{\psi}_1'\boldsymbol{\psi}_1/4N$ and also tends to zero. The first term is distributed as

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_k y_k^2, \tag{19}$$

where $y_1, y_2, \ldots, y_k$ are independent standard normal deviates and the λ_l are the latent roots of

$$\frac{1}{N}\mathbf{D}^{-1}\boldsymbol{\Lambda}_1 = \mathbf{I} - \mathbf{1}\mathbf{1}'\mathbf{D} - \mathbf{D}^{-1}\boldsymbol{\psi}_0\boldsymbol{\psi}_0'. \tag{20}$$

Since (20) can be written as the sum of the matrices

$$\mathbf{I} - \mathbf{1}\mathbf{1}'\mathbf{D} - \frac{\mathbf{D}^{-1}\boldsymbol{\psi}_0\boldsymbol{\psi}_0'}{\boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_0}, \quad (1 - \boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi})\frac{\mathbf{D}^{-1}\boldsymbol{\psi}_0\boldsymbol{\psi}_0'}{\boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_0},$$

the first of which is idempotent of rank $k-2$, the second of which is a multiple, $1 - \boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_0$, of an idempotent with a zero product with the first matrix, the roots λ_l of (20) are

$$1 - \boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_0, \quad 1 \ (k-2 \text{ times}). \tag{21}$$

Thus the distribution of X^2 is here

$$\chi^2_{k-2} + (1 - \boldsymbol{\psi}_0' \mathbf{D}^{-1} \boldsymbol{\psi}_0) y_1^2. \tag{22}$$

The root $1 - \boldsymbol{\psi}_0' \mathbf{D}^{-1} \boldsymbol{\psi}_0$ is shown in Tables 1 and 2 for various values of k and types of class intervals. It lies between 0 and 1 and tends to zero as $k \to \infty$ provided the probability content of all the classes tends to zero. A heuristic proof of this limit may easily be given since

$$\begin{aligned} \boldsymbol{\psi}_0' \mathbf{D}^{-1} \boldsymbol{\psi}_0 &= \sum_{l=1}^{k} \frac{\psi_0^2(l)}{p_l}, \\ &\sim \int_{-\infty}^{\infty} \frac{(d\phi)^2}{\phi\, dz}, \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{\phi} \frac{d\phi}{dz} \right)^2 \phi\, dz, \\ &= 1. \end{aligned} \tag{23}$$

To study the effect of using the mean m of an independent sample of N, instead of $\bar{x}$, to estimate μ, consider new variables $t_i = x_i - m$. They are jointly normal with zero means, variances $1 + 1/N$ and correlations $1/(N+1)$. Here n_l will be the number of t_i which fall in (z_{l-1}, z_l). Introducing suitable indicators to achieve this, the previous argument may be repeated and it leads to

$$\left.\begin{aligned} E(n_l) &= N p_l - \tfrac{1}{2} \psi_1(l), \\ \operatorname{var}(n_l) &= N(p_l + p_l^2 + \psi_0^2(l)), \\ \operatorname{covar}(n_l, n_m) &= N(-p_l p_m + \psi_0(l)\, \psi_0(m)). \end{aligned}\right\} \tag{24}$$

Thus, in this case, X^2 is distributed as

$$\chi^2_{k-2} + (1 + \boldsymbol{\psi}_0' \mathbf{D}^{-1} \boldsymbol{\psi}_0) y_1^2, \tag{25}$$

which, for large k, is equal to a $\chi^2_{k-2} + 2y_1^2$.

The results (22), (25) and the standard result when μ is known may be written together, when k is large, as

$$X^2 = \begin{cases} \chi^2_{k-2} + 2y_1^2 & \text{(mean estimated from an independent sample of } N\text{)}, \\ \chi^2_{k-1} & \text{(mean known)}, \\ \chi^2_{k-2} & \text{(mean estimated from sample)}. \end{cases} \tag{26}$$

The results (26) could be interpreted roughly to mean that the correlation (or the conformity of the sample to the hypothesis) from using the sample mean is 'worth' two degrees of freedom in overcoming the effect of variability which is 'worth' one degree of freedom.

4. Normal distribution, mean and variance unknown

First it will be supposed that the mean and variance of the normal distribution are estimated by $\bar{x}$ and s^2, found from the sample under test. In this case $Z_1, Z_2, \ldots, Z_{k-1}$ will be found from

$$p_l = \int_{Z_{l-1}}^{Z_l} \exp\left\{ \frac{-(t-\bar{x})^2}{2s^2} \right\} \frac{dt}{s\sqrt{(2\pi)}},$$

so that $Z_l = \bar{x} + z_l s$. Thus $Z_{l-1} \leqslant x_i \leqslant Z_l$ is the same as

$$\frac{Z_{l-1}-\bar{x}}{s} \leqslant \frac{x_i-\bar{x}}{s} \leqslant \frac{Z_l-\bar{x}}{s},$$

i.e.

$$z_{l-1} \leqslant \frac{x_i-\bar{x}}{s} \leqslant z_l.$$

The variates $t_i = (x_i-\bar{x})/s$ have an awkward distribution with a finite range

$$(-\sqrt{N}, \sqrt{N})\,(1+(N-1)^{-2})^{-\frac{1}{2}}.$$

By an argument, often used for serial correlations (see, for example, Watson (1955)),

$$E(t_1^a t_2^b) = \frac{E\{(x_1-\bar{x})^a (x_2-\bar{x})^b\}}{E(s^{a+b})},$$

so that all the joint moments of t_1 and t_2 are avail ole. An asymptotic expansion for $f(t_1, t_2)$ can be found by writing (with $H_i(x)$, the ith Hermite polynomial as defined by Cramér (1946))

$$f(t_1, t_2) = \phi(t_1)\,\phi(t_2)\left(\sum_{j=0}^{\infty}\sum_{i=0}^{\infty} a_{ij} H_i(t_1)\,H_j(t_2)\right)$$

and evaluating the coefficients a_{ij} up to and including those of order N^{-1}. Since

$$f(t_1, t_2) = f(t_2, t_1),$$

we have $a_{ij} = a_{ji}$. The first four moments of t_1 (and t_2) are

$$0, \quad 1, \quad 0, \quad 3\left(1-\frac{2}{N+1}\right),$$

while

$$\mu_{11} = \frac{-1}{N-1}, \quad \mu_{22} = \left(1+\frac{2}{(N-1)^2}\right)\left(1-\frac{2}{N+1}\right).$$

Hence (to order required)

$$a_{00} = 1, \qquad a_{04} = -1/4N,$$
$$a_{11} = -1/N, \quad a_{22} = -1/2N;$$
$$a_{ij} = 0, \quad \text{otherwise},$$

as may be shown by further calculations, so that

$$f(t_1, t_2) = \phi(t_1)\,\phi(t_2)\left\{1 - \frac{H_1(t_1)\,H_1(t_2)}{N} - \frac{H_2(t_1)\,H_2(t_2)}{2N} - \frac{H_4(t_1)+H_4(t_2)}{4N}\right\}. \tag{27}$$

To apply the method of § 2, indicators $Y_i(l)$ are defined as in (11) but using $(x_i-\bar{x})/s$ instead of $x_i-\bar{x}$. From (27), it follows that

$$P(Y_1(l)=1) = \Phi_0 - \frac{1}{4N}(\Phi_4 - 6\Phi_2 + 3\Phi_0),$$

$$P(Y_1(l)\,Y_2(m)=1) = \Phi_0(l)\,\Phi_0(m) - \frac{\Phi_1(l)\,\Phi_1(m)}{N} - \frac{(\Phi_2(l)-\Phi_0(l))\,(\Phi_2(m)-\Phi_0(m))}{2N}$$
$$- \frac{(\Phi_4(l)-6\Phi_2(l)+3\Phi_0(l))\,\Phi_0(m) + \Phi_0(l)\,(\Phi_4(m)-6\Phi_2(m)+3\Phi_0(m))}{4N}.$$

Introducing these formulae into (6), (7), (8) and using (16) to simplify them, we find that $E(n_l)$ differs from Np_l by terms of $O(N^{-1})$ so that the last term in (9) vanishes as $N \to \infty$, and that the covariance matrix of the n_l is defined by

$$\text{var}\,(n_l) = N(p_l - p_l^2 - \psi_0^2(l) - \tfrac{1}{2}\psi_1^2(l)),$$

$$\text{covar}\,(n_l, n_m) = N(-p_l p_m - \psi_0(l)\,\psi_0(m) - \tfrac{1}{2}\psi_1(l)\,\psi_1(m)).$$

Thus, in matrix form, the covariance matrix may be written as

$$\mathbf{\Lambda}_2 = N(\mathbf{D} - \mathbf{D}\mathbf{1}\mathbf{1}'\mathbf{D} - \boldsymbol{\psi}_0\boldsymbol{\psi}_0' - \tfrac{1}{2}\boldsymbol{\psi}_1\boldsymbol{\psi}_1'), \tag{28}$$

which is the same as (18) except for the additional term in $\boldsymbol{\psi}_1$.

Thus the limiting distribution of X^2 is, in this case, of the form (12) where the λ_l are the latent roots of

$$\frac{1}{N}\mathbf{D}^{-1}\mathbf{\Lambda}_2 = \mathbf{I} - \mathbf{1}\mathbf{1}'\mathbf{D} - \mathbf{D}^{-1}\boldsymbol{\psi}_0\boldsymbol{\psi}_0' - \tfrac{1}{2}\mathbf{D}^{-1}\boldsymbol{\psi}_1\boldsymbol{\psi}_1'. \tag{29}$$

The latent roots λ_l of (29) are the same as those of

$$\mathbf{I} - \mathbf{D}^{\frac{1}{2}}\mathbf{1}\mathbf{1}'\mathbf{D}^{\frac{1}{2}} - \mathbf{D}^{-\frac{1}{2}}\boldsymbol{\psi}_0\boldsymbol{\psi}_1'\mathbf{D}^{-\frac{1}{2}} - \tfrac{1}{2}\mathbf{D}^{-\frac{1}{2}}\boldsymbol{\psi}_1\boldsymbol{\psi}_1'\mathbf{D}^{-\frac{1}{2}}$$

so that the $\mu_l = 1 - \lambda_l$ are the roots of

$$\mathbf{D}^{\frac{1}{2}}\mathbf{1}\mathbf{1}'\mathbf{D}^{\frac{1}{2}} + \mathbf{D}^{-\frac{1}{2}}\boldsymbol{\psi}_0\boldsymbol{\psi}_0'\mathbf{D}^{-\frac{1}{2}} + \tfrac{1}{2}\mathbf{D}^{-\frac{1}{2}}\boldsymbol{\psi}_1\boldsymbol{\psi}_1'\mathbf{D}^{-\frac{1}{2}},$$

i.e.

$$[\mathbf{D}^{\frac{1}{2}}\mathbf{1},\ \mathbf{D}^{-\frac{1}{2}}\boldsymbol{\psi}_0,\ 2^{-\frac{1}{2}}\mathbf{D}^{-\frac{1}{2}}\boldsymbol{\psi}_1]\begin{bmatrix}\mathbf{1}'\mathbf{D}^{\frac{1}{2}}\\ \boldsymbol{\psi}_0'\mathbf{D}^{-\frac{1}{2}}\\ 2^{-\frac{1}{2}}\boldsymbol{\psi}_1'\mathbf{D}^{-\frac{1}{2}}\end{bmatrix}.$$

But, by a well-known matrix result, the non-zero latent roots of $\mathbf{A}\mathbf{A}'$, where $\mathbf{A}$ is an arbitrary matrix, are the same as those of $\mathbf{A}'\mathbf{A}$. Since

$$\mathbf{1}'\mathbf{D}\mathbf{1} = 1, \quad (\mathbf{1}'\mathbf{D}^{\frac{1}{2}})(\mathbf{D}^{-\frac{1}{2}}\boldsymbol{\psi}_0) = \mathbf{1}'\boldsymbol{\psi}_0 = 0, \quad (\mathbf{1}'\mathbf{D}^{\frac{1}{2}})(\mathbf{D}^{-\frac{1}{2}}\boldsymbol{\psi}_1) = \mathbf{1}'\boldsymbol{\psi}_1 = 0,$$

the μ_l are the latent roots of

$$\begin{bmatrix}1, & 0, & 0\\ 0, & \boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_0, & 2^{-\frac{1}{2}}\boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_1\\ 0, & 2^{-\frac{1}{2}}\boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_1, & \tfrac{1}{2}\boldsymbol{\psi}_1'\mathbf{D}^{-1}\boldsymbol{\psi}_1\end{bmatrix}. \tag{30}$$

Thus, the roots λ_l of (29) are 1 ($k-3$ times), $1-\mu_1$, $1-\mu_2$, where μ_1 and μ_2 are the roots of (30).

For arbitrary systems of class intervals $\boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_1$ is not zero and the quadratic equation

$$\begin{vmatrix}\boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_0 - \mu & \boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_1\\ \boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_1 & \boldsymbol{\psi}_1'\mathbf{D}^{-1}\boldsymbol{\psi}_1 - 2\mu\end{vmatrix} = 0$$

must be solved. If, however, the intervals $(-\infty, Z_1), (Z_1, Z_2), \ldots, (Z_{k-1}, \infty)$ are symmetrically placed with respect to the sample mean—or the intervals $(-\infty, z_1), (z_1, z_2), \ldots, (z_{k-1}, \infty)$ symmetrically placed with respect to zero—inspection shows that $\boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_1$ is always zero. In this case, the roots of $(ND^{-1})\mathbf{\Lambda}_2$ are therefore

$$\lambda_1 = (1 - \boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_0), \quad \lambda_2 = (1 - \tfrac{1}{2}\boldsymbol{\psi}_1'\mathbf{D}^{-1}\boldsymbol{\psi}_1), \quad \lambda_l = 1 \quad (k-3 \text{ times}). \tag{31}$$

Since the second term of (9) vanishes stochastically as $N \to \infty$, we have the result that the limiting distribution of X^2 is that of

$$\chi^2_{k-3} + \lambda_1 y_1^2 + \lambda_2 y_2^2. \tag{32}$$

It is shown in the tables that λ_2 tends to zero as $k \to \infty$, although less rapidly than λ_1.

Finally, to see whether the effect of using an estimator is the same for the variance as for the mean, we consider the case where the sample mean and an estimator of variance with $N-1$ degrees of freedom, from an independent sample, are used. In this case

$$t_i = \frac{x_i - \bar{x}}{s\sqrt{(1-1/N)}} \quad (i = 1, 2)$$

have, as has been shown by Cornish (1954), the joint density

$$\frac{\Gamma[\frac{1}{2}(N+1)]\,(1-\rho^2)^{-\frac{1}{2}}}{\Gamma[\frac{1}{2}(N-1)]\,\pi(N-1)} \left(1 + \frac{Q}{N-1}\right)^{-\frac{1}{2}(N+1)},$$

where $\rho = -1/(N-1)$, $Q = (t_1^2 - 2\rho t_1 t_2 + t_2^2)/(1-\rho^2)$. To $O(N^{-1})$

$$f(t_1, t_2) = \frac{e^{-\frac{1}{2}Q}}{2\pi(1-\rho^2)^{\frac{1}{2}}} + \frac{1}{N}\frac{e^{-\frac{1}{2}Q}}{2\pi(1-\rho^2)^{\frac{1}{2}}}(-Q + \tfrac{1}{4}Q^2) \tag{33}$$

so that, defining new indicators $Y_i(l)$ in the obvious way,

$$P(Y_1(l)\,Y_2(m) = 1) = P\{Y_1(l)\,Y_2(m) = 1, \text{ when } \sigma^2 \text{ is known}\} + \frac{1}{N}\int_{z_{l-1}}^{z_l}\int_{z_{m-1}}^{z_m} \frac{e^{-\frac{1}{2}(t_1^2+t_2^2)}}{2\pi}\{-(t_1^2+t_2^2) + \tfrac{1}{4}(t_1^2+t_2^2)^2\}\,dt_1\,dt_2,$$

where, in the integral, N has been put equal to ∞ since this term is already $O(N^{-1})$. Thus

$$P(Y_1(l)\,Y_2(m) = 1) = P\{Y_1(l)\,Y_2(m) = 1, \text{ when } \sigma^2 \text{ is known}\} + \frac{1}{N}\left\{-\Phi_2(l)\,\Phi_0(m) - \Phi_0(l)\,\Phi_2(m) + \frac{\Phi_4(l)\,\Phi_0(m) + 2\Phi_2(l)\,\Phi_2(m) + \Phi_0(l)\,\Phi_4(m)}{4}\right\}. \tag{34}$$

Since

$$f(t_1) = \phi(t) + \frac{1}{N}\phi(t)\left\{-t^2 - 1 + \frac{t^4 + 2t^2 + 3}{4}\right\},$$

$$P(Y_1(l) = 1) = \Phi_0 + \frac{1}{N}\left\{-\Phi_2 - \Phi_0 + \frac{\Phi_4 + 2\Phi_2 + 3\Phi_0}{4}\right\}. \tag{35}$$

Introducing (34) and (35) into (6), (7), (8) and using the expansions (16), the covariance matrix of the n_l is here found to be

$$\mathbf{\Lambda}_3 = \mathbf{I} - \mathbf{1}\mathbf{1}'\mathbf{D} - \mathbf{D}^{-1}\boldsymbol{\psi}_0\boldsymbol{\psi}_0' + \tfrac{1}{2}\mathbf{D}^{-1}\boldsymbol{\psi}_1\boldsymbol{\psi}_1'.$$

Following the previous method, X^2 has as its limiting distribution

$$\chi^2_{k-3} + \lambda_1 y_1^2 + \lambda_3 y_3^2, \tag{36}$$

where

$$\lambda_1 = 1 - \boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_0, \quad \lambda_3 = 1 + \tfrac{1}{2}\boldsymbol{\psi}_1'\mathbf{D}^{-1}\boldsymbol{\psi}_1, \tag{37}$$

provided $\boldsymbol{\psi}_0'\mathbf{D}^{-1}\boldsymbol{\psi}_1 = 0$, for which symmetry of the intervals (z_{l-1}, z_l) about zero is a sufficient condition. Since $\lambda_3 \to 2$, as all the intervals tend to zero, we have then

$$X^2 = \begin{cases} \chi^2_{k-3} + 2y_3^2 & \text{(variance estimated independently of the sample),} \\ \chi^2_{k-2} & \text{(variance known),} \\ \chi^2_{k-3} & \text{(variance estimated from the sample).} \end{cases} \tag{38}$$

In each case of (38) it is assumed that the sample mean has been used. Comparing (38) with (26), it is seen that the variance behaves in the same way as the mean, in this regard.

5. NUMERICAL RESULTS

Only the practical case where the mean and variance have been estimated by the sample mean and variance, will now be considered. Restating the results (31), (32), for the simple case of symmetrically placed intervals, we have

$$X^2 \sim \chi^2_{k-3} + \lambda_1 y_1^2 + \lambda_2 y_2^2,$$

where
$$\lambda_1 = 1 - \sum_{l=1}^{k} \frac{\psi_0^2(l)}{p_l}, \quad \lambda_2 = 1 - \frac{1}{2} \sum_{l=1}^{k} \frac{\psi_1^2(l)}{p_l}.$$

and ψ_0, ψ_1 are defined in (15).

In Table 1, the values of λ_1 and λ_2, $E(X^2)$ and $k-3$ are given for various k when class intervals of equal probability content are used, in the spirit of Mann & Wald (see § 1).

In Table 2, the same quantities are given for the rule of § 1 using intervals of equal length. A further column gives h, the length of the interval (z_{l-1}, z_l), for $l = 2, \ldots, k-1$.

It is now of interest to compare our results with those of Chernoff & Lehmann. As described in § 1 they used a *fixed* system of class intervals $(-\infty, Z_1)$, (Z_1, Z_2), ..., (Z_{k-1}, ∞) and a normal

Table 1

k	λ_1	λ_2	$k-3$	$E(X^2)$
2	0·363	1	−1	0·363
3	0·207	0·773	0	0·980
4	0·139	0·619	1	1·758
5	0·103	0·592	2	2·695
6	0·081	0·459	3	3·540
10	0·042	0·297	7	7·339
$\to\infty$	0	0	$k-3$	$k-3$

Table 2

k	h	λ_1	λ_2	$k-3$	$E(X^2)$
2	∞	0·363	1·000	−1	0·363
3	1·0	0·197	0·738	0	0·935
4	1·0	0·118	0·459	1	1·577
4	0·5	0·174	0·738	1	1·912
4	0·4	0·201	0·797	1	1·998
5	1·0	0·088	0·272	2	2·360
5	0·5	0·116	0·584	2	2·700
5	0·4	0·145	0·676	2	2·821
6	1·0	0·080	0·184	3	3·264
6	0·5	0·077	0·440	3	3·517
10	1·0	0·077	0·145	7	7·222
10	0·5	0·025	0·103	7	7·128
$\to\infty$	—	0	0	$k-3$	$k-3$

population with mean μ and variance σ^2. Their result is of the same form as ours and λ_1 and λ_2 are again the roots of a certain matrix. If $Z_1, Z_2, \ldots, Z_{k-1}$ are expressed in terms of $z_1, z_2, \ldots, z_{k-1}$ by the relations

$$Z_l = \mu + z_l \sigma \quad (l = 1, \ldots, k-1),$$

their results should be comparable with ours when one sets $\mu = 0$, $\sigma = 1$ in the matrix given by Chernoff & Lehmann for the roots μ_l ($\lambda_l = 1 - \mu_l$). In fact, the matrices are then *identical*. This is also seen arithmetically in Table 2; the values of λ_1 and λ_2 for $k = 4$, $h = 0{\cdot}4$ are the same as those in Chernoff & Lehmann's example where the intervals are $(-\infty, -1)$, $(-1, 0)$, $(0, 1)$, $(1, \infty)$ and $\mu = 0$, $\sigma^2 = 2{\cdot}5$.

This identity of our results with those of Chernoff & Lehmann when $\mu = 0$, $\sigma^2 = 1$ has a simple interpretation. It means that, asymptotically, there is no difference between using (μ, σ^2) and $(\bar{x}, s^2)$ for positioning and scaling the class intervals. Thus the variation of the class intervals from sample to sample has no effect on the distribution of X^2. The reason why X^2 here is not distributed as χ^2_{k-3} is the same as that pointed out by Chernoff & Lehmann —the use of estimators based on the sample values and not on the cell frequencies. If this is generally true, Chernoff & Lehmann's general theorem will hold whenever the class intervals are conditioned by the values of the sample estimators.

The question, which now arises, is how the class intervals should be chosen. In § 1, it was seen that considerations of power have led to the suggestion that many more intervals be used than now seems to be customary. Williams (1950) has suggested that Mann–Wald optimum is a broad one and that little will be lost if their recommended number of classes is halved. Our results, especially Tables 1 and 2, suggest that, if X^2 is to be compared with the significance points of χ^2_{k-3}, the number of classes should be large to avoid underestimation of the significance level. Before any specific recommendation can be made, agreement must be reached on the precise amount of underestimation which can be tolerated. However, any such judgement seems arbitrary and it seems better to give a method whereby anyone can derive his own rule.

The asymptotic distribution of X^2 is that of a quadratic form in normal variables with a dominant part χ^2_{k-3} and small part $\lambda_1 y_1^2 + \lambda_2 y_2^2$. The method of Pitman & Robbins (1949) is not useful for a quadratic form of this type. But a variant* of the method of Harper & Macdonald (1957) is ideally suitable. When k is not much smaller than 10 the probability that X^2 is greater than u is given accurately by

$$\frac{e^{-\frac{1}{2}u} u^{\frac{1}{2}k-1}}{\Gamma(\frac{1}{2}k)\, 2^{\frac{1}{2}k-1} \sqrt{[(1-\lambda_1)(1-\lambda_2)]}} \left\{1 + \frac{a}{u} + \frac{a_2}{u^2}\right\}, \tag{39}$$

where

$$\left.\begin{aligned} a_1 &= (2 - \tfrac{1}{2}(\nu_1 + \nu_2)(\tfrac{1}{2}k - 1), \\ a_2 &= [4 + \tfrac{3}{8}(\nu_1^2 + \nu_2^2) - (\nu_1 + \nu_2) + \tfrac{1}{4}\nu_1\nu_2](\tfrac{1}{2}k - 1)(\tfrac{1}{2}k - 2), \\ \nu_1 &= 2\lambda_1/(1-\lambda_1), \\ \nu_2 &= 2\lambda_2/(1-\lambda_2). \end{aligned}\right\} \tag{40}$$

Thus, a given system of intervals may be examined by first computing λ_1 and λ_2 from the formulae at the beginning of this section and then using (39) and (40) to find the probability that X^2 exceeds the 5 % (say) tabular value of χ^2_{k-3}. Depending on what is considered to be a tolerable deviation of this probability from 0·05, the system of intervals will be judged as satisfactory or not.

* Suggested to the author in a letter from J. A. Macdonald.

It is clear that no hard and fast rule will find acceptance, but it does seem worth while to suggest a working rule of the very simplest kind. Ten intervals is a convenient round number of intervals. Then the probability of X^2 exceeding $\chi^2_7(5\,\%) = 14{\cdot}067$ may be found, from (39) and (40), to be 0·057 for the case $k = 10$ in Table 1 and to be 0·054 and 0·052 for the cases $k = 10$, $h = 1, 0{\cdot}5$ (respectively) in Table 2. Thus, if ten intervals are formed from almost any kind of subdivision of $(-\infty, \infty)$, one can be sure that true significance level is between 0·05 and 0·06—and usually much nearer to 0·05. Our rule is therefore to use at least ten intervals, none of which should have 'small' expected frequencies. This latter provision is made to reduce the possibility of small sample effects invalidating the asymptotic results on which the rule rests.

The methods developed above have a bearing on a slightly different kind of problem, brought to my attention by J. B. Douglas. J. F. McIllwraith of the Storm Water Standards Committee, Sydney, examined the primary rain falls at 142 different stations to see whether the hypothesis of log-normal rain falls was tenable. A goodness-of-fit X^2 was calculated from the results at each station, six equally probable intervals being used so that the cell boundaries were given by

$$-\infty, \quad \bar{x} - 0{\cdot}9674s, \quad \bar{x} - 0{\cdot}4307s, \quad 0, \quad \bar{x} + 0{\cdot}4307s, \quad \bar{x} + 0{\cdot}9674s, \quad +\infty.$$

The mean X^2 was 3·80 and Douglas observed that, because of the work of Chernoff & Lehmann, this need not imply that the hypothesis should be rejected. With the results of the present paper, the matter may be taken a step further since these X^2 have been calculated exactly on the assumptions of this paper. In particular, from Table 1, $k = 6$ gives $\lambda_1 = 0{\cdot}081$, $\lambda_2 = 0{\cdot}459$, so that on the null hypothesis, these X^2 are distributed as

$$\chi^2_3 + 0{\cdot}081y_1^2 + 0{\cdot}459y_2^2.$$

Thus

$$E(X^2) = 3{\cdot}540,$$

$$\text{var}\,(X^2) = 6{\cdot}217.$$

The mean of 142 values of X^2 will be approximately normal with mean 3·540 and variance 0·04378. Since

$$\frac{3{\cdot}80 - 3{\cdot}54}{\sqrt{0{\cdot}04378}} = 1{\cdot}24,$$

the data, assuming homogeneity and independence of stations, do not refute the hypothesis.

The author is grateful to Prof. P. A. P. Moran and Dr E. J. Hannan for some helpful discussions, to Mr J. A. Macdonald for providing his unpublished method of deriving the distribution of X^2, to Mr J. B. Douglas and Mr J. F. McIllwraith for providing the data in §5, and to the Editor and the Referee for the references to previous discussions of this problem.

[*Editorial Note added in proof.* The Editor unfortunately failed to draw the author's attention to a paper by D. E. Barton, entitled 'Neyman's Ψ_k^2 test of goodness of fit when the null hypothesis is composite', read at the International Congress of Mathematicians, Amsterdam (1954), and published in the *Skandinavisk Actuarietidskrift* (1956), pp. 216–45. Taking the rather more general Ψ^2 statistic, which becomes X^2 in a special case, Barton considered the use of variable class intervals and reached a conclusion very similar to Watson's. Before going to press it has not been possible to ask Dr Watson to examine the relation of these earlier results to his work. E.S.P.]

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APPROXIMATIONS TO THE DISTRIBUTIONS OF SOME MEASURES OF DISPERSION BASED ON SUCCESSIVE DIFFERENCES

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1. Introduction

The usual measures of dispersion such as the sample variance, the sample mean deviation, the sample range, etc. cannot be used when the mean of the population is undergoing a trend, since they are likely to be seriously affected by the trend. Under these circumstances and especially if the trend is a slow-moving one the following four measures of dispersion based on successive differences have been proposed:

$$\delta^2 = \frac{1}{n-1}\sum_{i=1}^{n-1}(x_i - x_{i+1})^2, \tag{1}$$

$$d = \frac{1}{n-1}\sum_{i=1}^{n-1}|x_i - x_{i+1}|, \tag{2}$$

$$\delta_2^2 = \frac{1}{n-2}\sum_{i=1}^{n-2}(x_i - 2x_{i+1} + x_{i+2})^2, \tag{3}$$

$$d_2 = \frac{1}{n-2}\sum_{i=1}^{n-2}|x_i - 2x_{i+1} + x_{i+2}|. \tag{4}$$

The theory of the distribution of these statistics and their usefulness has been discussed in some detail by Von Neumann, Kent, Bellinson & Hart (1941) and Kamat (1953*a*, *b*, 1954) among others. These measures of dispersion are considerably less efficient as estimators than the sample variance, s^2, when the mean of the parent population remains constant. But s^2 suffers from a heavy bias if the mean is undergoing a trend while statistics based on the successive differences are comparatively unaffected.

It is not easy, however, to obtain the distribution of these statistics. Except for a modified form of δ^2 (for which see Kamat, 1955), it has not yet been possible to find the exact distributions of these statistics. The procedure so far has been to approximate their distributions by appropriate Pearson-type curves based on their first three or four moments. Apart from the laborious computation involved in the fitting of Pearson curves to these measures of dispersion, a serious drawback to these approximations is that they cannot be used readily for comparing two independent estimates of variability about the trend taken over two different periods of time.

Recently Cadwell (1953, 1954) has successfully used a *power of* χ^2 to obtain approximate distributions to some measures of dispersion such as the mean deviation, the mean range and others. Following this method, one assumes that a statistic u is approximately distributed as $(\chi_\nu^2/c)^\alpha$ or, taking $\lambda = 1/\alpha$, that cu^λ is approximately distributed as χ^2 with ν degrees of freedom. The constants c, α (or λ) and ν are then determined by equating the first three moments.

In this paper we have used this method to obtain approximations to the distributions of each of the four statistics mentioned above. Our results show that over a wide range of sample size the same power of χ^2, i.e. a constant value for α or λ, can be used for each of

these statistics. Consequently, it not only becomes easy to find approximate percentage points by using percentage points of χ^2, but it is also now possible to compare two different estimates of variability by use of the F test. This is because we may write

$$F = \frac{c_1 \nu_2}{c_2 \nu_1} \left(\frac{u_1}{u_2}\right)^{\lambda}, \tag{5}$$

where c_1, c_2, and ν_1, ν_2 are the scale parameters and equivalent degrees of freedom, found from Table 1, of the two u estimators to be compared and λ is the common power.

Table 1. *Values of ν and $\log c$ for measures of dispersion based on successive difference, keeping λ constant*

n	δ^2/σ^2 $\lambda = 0{\cdot}7770$		d/σ $\lambda = 1{\cdot}3219$		δ_2^2/σ^2 $\lambda = 0{\cdot}7353$		d_2/σ $\lambda = 1{\cdot}2195$	
	ν	$\log_{10} c$	ν	$\log_{10} c$	ν	$\log_{10} c$	ν	$\log_{10} c$
5	4·879	0·4775	6·053	0·6957	3·528	0·0135	4·562	0·0288
6	5·978	0·5618	7·434	0·7880	—	—	—	—
7	7·081	0·6326	8·816	0·8642	5·398	0·1863	7·042	0·4822
8	8·182	0·6933	10·193	0·9288	—	—	—	—
9	9·284	0·7465	11·579	0·9853	—	—	—	—
10	10·39	0·7941	12·963	1·0353	8·24	0·3615	10·815	0·6720
11	11·49	0·8368	14·343	1·0800	—	—	—	—
12	12·60	0·8760	15·732	1·1208	—	—	—	—
13	13·70	0·9116	17·114	1·1579	—	—	—	—
14	14·80	0·9445	18·500	1·1921	—	—	—	—
15	15·91	0·9754	19·885	1·2240	12·99	0·5530	17·13	0·8744
16	17·01	1·0039	21·27	1·2535	—	—	—	—
17	18·12	1·0309	22·65	1·2811	—	—	—	—
18	19·23	1·0564	24·04	1·3072	—	—	—	—
19	20·32	1·0800	25·42	1·3317	—	—	—	—
20	21·43	1·1028	26·81	1·3550	17·74	0·6853	23·45	1·0119
25	26·94	1·2010	33·74	1·4557	22·50	0·7868	29·78	1·1164
30	32·47	1·2814	40·64	1·5370	27·26	0·8690	36·12	1·2007
40	43·52	1·4076	54·50	1·6651	36·77	0·9975	48·8	1·3319
50	54·56	1·5052	68·33	1·7637	46·28	1·0966	61·4	1·4320

The basis of Table 1 is described in § 3–7 below. It may be mentioned here that we have considered these statistics only for $n \geqslant 5$. The approximations are not expected to be adequate for lower sample sizes, especially if λ is taken as constant and, moreover, it is unlikely that statistics based on successive differences will be used for very small samples.

It seems probable that tests based on the mean-square differences, δ^2 or δ_2^2, will in general have somewhat greater power of discrimination than those based on the mean absolute differences, d and d_2.* The possible advantages in using the absolute differences are: (i) speed

* If the variation is random and there is no trend, the asymptotic efficiencies for estimating σ^2 are known to be as follows:

δ^2: 66·7%, δ_2^2: 51·4%; d: 60·5%, d_2: 47·1%.

in calculation, (ii) greater accuracy in the $(\chi^2_\nu/c)^\alpha$ approximation (see Tables 3, 6, 7 below), (iii) greater robustness if the variation is not normal.

2. EXAMPLES

Before proceeding to examine these approximations in more detail, we shall illustrate the use of the proposed variance-ratio test in comparing variability about the trend in a time series. The data used are dust readings taken with a Tyndallometer in a mine shaft. The readings were taken at half-minute intervals and the four series A, B, C and D were collected on the same morning during different periods, starting at the times shown. We are indebted to the Director of the Safety in Mines Research Establishment, Sheffield, for permission to use these data. The figures quoted are Tyndallometer readings multiplied by 1000, and the

Series A (starting at 10.04 hr.)	173	176	130	153	137	144	95	137	131	126
	167	118	99	140	111	81	83	106	104	117
	78	88	119	87	124	125	102	109	92	114
	85	110	100	—	—	—	—	—	—	—
Series B (starting at 10.20½ hr.)	70	28	52	49	36	38	45	43	38	57
	40	51	56	58	39	25	33	45	45	34
	43	42	20	34	18	30	23	23	19	30
Series C (starting at 10.47 hr.)	33	43	37	40	43	40	31	36	51	36
	45	43	36	74	79	69	85	95	128	136
Series D (starting at 10.57 hr.)	99	59	165	108	167	145	113	124	117	119
	162	187	115	186	160	106	142	110	153	165
	137	99	106	89	83	—	—	—	—	—

sequences in time read from left to right and from line to line. The four series are plotted in Fig. 1. It is clear that a trend is present in series A, B and C and poss ibly in D. For random series, it is known that*

$$\mathscr{E}(\delta^2) = 2\sigma^2; \quad \mathscr{E}(d) = 2\sigma/\sqrt{\pi} = 1{\cdot}1284\sigma; \quad \mathscr{E}(\delta_2^2) = 6\sigma^2; \quad \mathscr{E}(d_2) = 2\sqrt{3}\,\sigma/\sqrt{\pi} = 1{\cdot}9544\sigma.$$

The upper section in Table 2 gives the four mean successive differ e nces as defined in equations (1)–(4) above for each series. Below these are five estima tes of the residual standard deviation, σ, i.e.

$$\{\Sigma(x_i-\bar{x})^2/(n-1)\}^{\frac{1}{2}}, \quad \sqrt{(\tfrac{1}{2}\delta^2)}, \quad d/1{\cdot}1284, \quad \sqrt{(\tfrac{1}{6}\delta_2^2)} \quad \text{and} \quad d_2/1{\cdot}9544.$$

For series D the random fluctuations are so great that the removal of a linear or parabolic trend has scarcely reduced the estimate of variability. For the other thr ee series the removal of a linear trend has an appreciable effect and for Series B and C there is a further, if smaller, reduction after allowance is made for a parabolic trend in the estimate s based on δ_2^2 and d_2. We shall now use the variance-ratio approximation to compare the residual variation in (i) A and D, (ii) B and C. We have from equation (5),

$$\log F = \log c_1 - \log c_2 + \log(\nu_2/\nu_1) + \lambda \log(u_1/u_2), \tag{6}$$

where the parameters $\log c_1$, $\log c_2$, ν_1, ν_2 and λ derived from Table 1 are given in Table 2†,

* Values for d and d_2 have been taken from Kamat (1953*b*, p. 119) and (1954, p. 7), respectively.

† Interpolation in Table 1 was necessary for the case $n = 33$.

while u_1 and u_2 are the corresponding values of δ^2, d, δ_2^2 or d_2 for the two series compared, also taken from Table 2.

Comparison of A and D

		F	ν_1	ν_2	Comment on significance
Comparison based on	δ^2	2·05	26·94	35·78	Just significant at 2·5 % level
	d	1·78	33·74	44·80	Between 5 and 2·5 % levels
	δ_2^2	2·02	22·50	30·11	Between 5 and 2·5 % levels
	d_2	1·51	29·78	39·92	Just not significant at 10 % level

The analysis suggests that the residual variation in Series D is greater than in A. We note, however, two points: (*a*) that the tests based on the mean-square differences, in this case at any rate, give clearer verdicts on significance than those based on the corresponding mean absolute differences; (*b*) that nothing has been gained by using second-difference estimates.

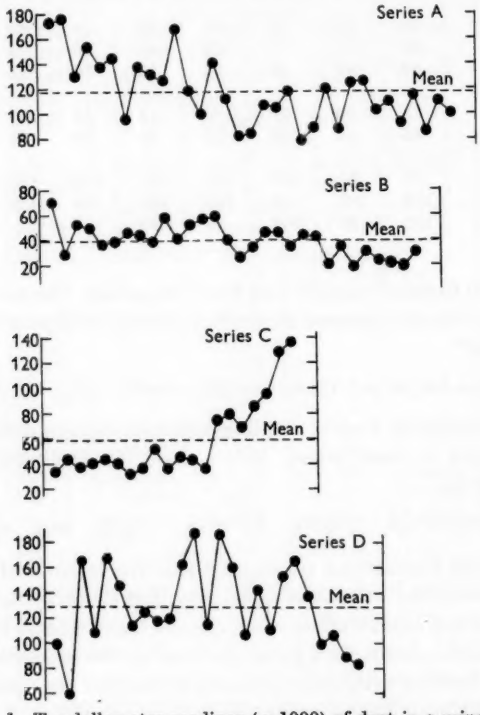

Fig. 1. Tyndallometer readings (× 1000) of dust in a mine shaft taken at half-minute intervals.

The first point is in accordance with what would be expected if the power of the former tests is the greater. With regard to (*b*), the chart for Series A suggests a curved trend, but this is perhaps not adequately represented by a second-order parabola.

Comparison of B and C

In this case we have only used δ^2 and δ_2^2; we find the following results:

		F	ν_1	ν_2	
Comparison based on	δ^2	1·05	21·43	32·47	Not significant
	δ_2^2	1·13	27·26	17·74	

The charts and standard deviation estimates given in Table 2 suggest that it is necessary to go at least to second differences before a legitimate comparison can be made, but clearly neither test provides any evidence for a difference in residual variation between the two series.

Table 2. *Data for comparison of Series A, B, C and D*

		A	B	C	D
	n	33	30	20	25
	δ^2	739·03	194·28	206·89	1852·25
	d	22·781	10·759	10·895	35·250
	δ_2^2	2282·35	501·04	418·44	5914·96
	d_2	43·355	17·750	16·889	61·043
Estimates of σ from	$\Sigma(x_i-\bar{x})^2$	26·32	12·85	31·35	33·39
	δ^2	19·22	9·85	10·17	30·43
	d	20·19	9·53	9·65	31·24
	δ_2^2	19·50	9·14	8·35	31·40
	d_2	21·67	9·08	8·64	31·23
Parameters from Table 1	For δ^2 λ	0·7770	0·7770	0·7770	0·7770
	ν	35·78	32·47	21·43	26·94
	$\log c$	1·3235	1·2814	1·1028	1·2010
	For d λ	1·3219	—	—	1·3219
	ν	44·80	—	—	33·74
	$\log c$	1·5798	—	—	1·4557
	For δ_2^2 λ	0·7353	0·7353	0·7353	0·7353
	ν	30·11	27·26	17·74	22·50
	$\log c$	0·9120	0·8690	0·6853	0·7868
	For d_2 λ	1·2195	—	—	1·2195
	ν	39·92	—	—	29·78
	$\log c$	1·2447	—	—	1·1164

3. Basic approximation

Cadwell (1953) has shown that if u is approximately distributed as $(\chi^2_\nu/c)^\alpha$, then ν and α are approximately given by the equations

$$\nu = \frac{2}{(3V-\sqrt{\beta_1})^2}\left\{1-\frac{(2V-\sqrt{\beta_1})^2(V+\sqrt{\beta_1})}{2(3V-\sqrt{\beta_1})}+\ldots\right\}, \tag{7}$$

$$\alpha = \alpha_0 - \frac{\alpha_0(\alpha_0-1)^2}{2\nu}+\ldots, \tag{8}$$

where $\alpha_0 = V\sqrt{(\frac{1}{2}\nu)}$, and V and β_1 are the values of the coefficient of variation and the β_1 constant of u. The constant c is then obtained from the equation

$$\log c = \log 2 + \lambda\{\log\Gamma(\alpha+\tfrac{1}{2}\nu) - \log\Gamma(\tfrac{1}{2}\nu) - \log\mathscr{E}(u)\}, \tag{9}$$

where $\lambda = \alpha^{-1}$. These relations have been used to find approximations to the distributions for each of the four statistics δ^2/σ^2, d/σ, δ_2^2/σ^2 and d_2/σ. The exact value of α was found by inverse interpolation making use of Brownlee's (1923) tables, and for small sample sizes double inverse interpolation was used to get the exact values of ν and α. (This has been outlined by Cadwell on p. 338 of his paper.) Unlike Cadwell we have retained one or more places of decimals in ν. In our case we find that by so doing it is easier to make the match of V and β_1 become more exact. It may be mentioned that the retaining of more places in ν does not involve appreciably more labour in the calculation of percentage points of these statistics, since in practice it was found that linear interpolation for ν in the percentage points of χ^2 is quite adequate for this purpose.

4. The mean-square successive difference: δ^2

Following the method summarized above and taking the moments from Von Neumann *et al.* (1941), ν, λ and $\log c$ were obtained for different values of n for the statistic δ^2/σ^2. Table 3 gives these values for a few representative sample sizes, $n = 5, 10, 20, 30, 50$. The fifth column gives the difference between the value of β_2 for the approximating distribution* and the true value of β_2; for $n > 5$ the difference is at the most in the second decimal place and it goes on decreasing as the sample size increases.

Table 3. *Approximation to* δ^2/σ^2 *by* $(\chi^2_\nu/c)^\alpha$

n	Variable λ				Fixed $\lambda = 0{\cdot}7770$	
	ν	λ	$\log c$	β_2 difference	β_1 difference	β_2 difference
5	4·98	0·7762	0·4788	+0·1007	−0·0013	+0·0913
10	10·40	0·7766	0·7946	+0·0380	+0·0001	+0·0368
20	21·4	0·7776	1·1020	+0·0163	+0·0003	+0·0180
30	32·5	0·7765	1·2819	+0·0123	+0·0001	+0·0113
50	54·4	0·7781	1·5036	+0·0050	+0·0003	+0·0070

* Following Cadwell (1953), we obtain β_2 and, later, β_1 values for the approximation by noting that for $x = (\frac{1}{2}\chi^2_\nu)$, the rth moment about zero is

$$\mu'_r = \Gamma(r\alpha+\tfrac{1}{2}\nu)/\Gamma(\tfrac{1}{2}\nu).$$

A variable value of λ does not permit us to construct tests corresponding to F tests. But the fuller table of values of λ (which is not reproduced above) indicated that a suitable fixed value of λ may be used as a compromise value for $n \geqslant 5$, thus allowing us the comparison of two independent estimates of variability based on δ^2. It was found that if we take $\lambda = 0{\cdot}7770$ it yields reasonably good approximation for $n \geqslant 5$. Table 1 gives the values of ν and $\log c$ for the fixed $\lambda = 0{\cdot}7770$ for $n = 5\,(1)\ 20,\ 25,\ 30,\ 40,\ 50$. In this case ν is calculated from the equation

$$\nu = \nu_0 + 0{\cdot}0824 - 0{\cdot}1436\,\nu_0^{-1} + \ldots, \tag{10}$$

where $\nu_0 = 3{\cdot}3127/V^2$, V^2 being the square of the coefficient of variation of δ^2. But this formula is useful only for $n \geqslant 9$, while for $n < 9$ the exact value of ν is obtained by inverse linear interpolation for the value of V^2, by using Brownlee's tables. $\log c$ was then evaluated from the relation (9) above.

We should expect that β_1 will not match exactly when λ is kept fixed. As mentioned above, in this case also we have retained two or three places of decimals in ν so as to match β_1 as closely as possible. The discrepancy between the β_1 value of the approximation and the true value of β_1 is now small, and it was found that it is only in the fourth decimal place for $n > 7$. Again, the discrepancy between the β_2 values of the approximation with a fixed λ and the true value of β_2 is of the same order as the discrepancy in these values resulting from a variable λ. Both these facts are illustrated by columns 6 and 7 of Table 3.

5. Comparison of approximate percentage points of δ^2/σ^2

The approximate percentage points of the distribution of δ^2/σ^2 can now be readily obtained by the use of the table of percentage points of the χ^2 distribution constructed by Hald & Sinkbæk (1950). Ordinary linear interpolation in these tables for fractional degrees of freedom is quite adequate to obtain the approximate significance points correct to two places of decimals.

Table 4. *Comparison of upper 5 % points given by various approximations to* δ^2/σ^2

Approximations	$n = 15$				$n = 20$			
	Lower		Upper		Lower		Upper	
	1%	5%	5%	1%	1%	5%	5%	1%
Moore (1)	0·49	0·77	3·70	4·71	0·63	0·90	3·45	4·27
Moore (2)	0·56	0·80	3·72	4·78	0·68	0·93	3·46	4·32
$(\chi^2_\nu/c)^\alpha$, α fixed	0·53	0·79	3·71	4·78	0·66	0·92	3·46	4·32
Pearson type VI	0·54	0·79	3·71	4·78	0·67	0·92	3·46	4·32

Various approximations have been recently suggested for the distribution of the mean-square successive difference. For instance, Moore (1955) has suggested two approximations: (1) $\delta^2/\sigma^2 \sim c\chi^2/\nu$, and (2) $\delta^2/\sigma^2 \sim c_1\chi^2/\nu + c_2$. The first is obtained by matching the first two moments and the second by matching the first three moments. Gayen & Jogdeo (1955) have

followed the method suggested by Cornish & Fisher (1937) of using the exact distribution of the sample variance s^2 as a first approximation to that of δ^2, and utilizing the first four cumulants of s^2 and δ^2 for obtaining approximations of higher order. We give in Table 4 comparisons of percentage points of δ^2/σ^2 for $n = 15$ and 20 obtained from the two approximations given by Moore (named Moore (1) and Moore (2)), the approximation $(\chi^2_\nu/c)^\alpha$ for fixed α suggested here and the Pearson type VI approximation based on the first four moments. Of these the last may be considered the closest to the true distribution of δ^2/σ^2 as it is based on

Table 5. *Percentage points for the approximate distribution of* δ^2/σ^2 (λ *constant*)

n	Lower				Upper			
	0·5	1·0	2·5	5	5	2·5	1·0	0·5
5	0·07	0·11	0·18	0·27	5·24	6·35	7·84	8·98
6	0·11	0·16	0·25	0·35	4·91	5·87	7·13	8·09
7	0·16	0·21	0·31	0·43	4·66	5·50	6·61	7·44
8	0·20	0·26	0·37	0·49	4·46	5·21	6·20	6·95
9	0·24	0·31	0·42	0·55	4·30	4·99	5·88	6·55
10	0·28	0·35	0·49	0·60	4·16	4·80	5·62	6·23
11	0·32	0·39	0·53	0·65	4·04	4·64	5·40	5·97
12	0·36	0·43	0·56	0·69	3·94	4·50	5·21	5·74
13	0·39	0·46	0·59	0·73	3·86	4·38	5·05	5·54
14	0·42	0·50	0·63	0·76	3·78	4·28	4·91	5·37
15	0·45	0·53	0·66	0·79	3·71	4·18	4·78	5·22
16	0·48	0·56	0·69	0·82	3·65	4·10	4·67	5·09
17	0·51	0·59	0·72	0·85	3·60	4·03	4·57	4·97
18	0·53	0·61	0·75	0·88	3·54	3·96	4·48	4·86
19	0·56	0·64	0·77	0·90	3·50	3·90	4·40	4·77
20	0·58	0·66	0·79	0·92	3·46	3·84	4·32	4·68
25	0·68	0·76	0·89	1·02	3·29	3·61	4·03	4·32
30	0·76	0·84	0·97	1·09	3·16	3·45	3·81	4·08
40	0·88	0·96	1·08	1·20	2·99	3·23	3·53	3·74
50	0·98	1·01	1·17	1·27	2·88	3·09	3·34	3·53

the first four moments, while all others are based on three moments. (The percentage points obtained from the approximation suggested by Gayen & Jogdeo (1955) are not given here as they do not appear to be as close to the Pearson approximation as the values obtained by the other three methods.)

It appears from this comparison that the approximation $(\chi^2_\nu/c)^\alpha$ is quite close to the Pearson-type approximation. The approximation, Moore (2), based on the first three moments is equally close; however, F tests based on it will not be as simple as those based on $(\chi^2_\nu/c)^\alpha$ for a fixed α. The approximation, Moore (1), which would be ideal for constructing F tests does not appear to be sufficiently close in the tails of the distribution.

We have not seen in the relevant papers on δ^2 any table of its approximate percentage points. We are therefore giving in Table 5 the approximate upper and lower 0·5, 1·0, 2·5 and 5·0 % points for δ^2/σ^2 for $n = 5(1)$ 20, 25, 30, 40 and 50 based on $(\chi^2_\nu/c)^\alpha$, α fixed.

6. THE MEAN SUCCESSIVE DIFFERENCE

In the same manner using the moments given by Kamat (1953*b*, p. 119), ν, λ and log c were obtained for the mean successive difference d/σ. A compromise value for a common λ was then found as $\lambda = 1{\cdot}3219$, and for this fixed value of λ, ν was calculated from the formula

$$\nu = \nu_0 + 0{\cdot}0593 - 0{\cdot}0610\,\nu_0^{-1} + \ldots, \qquad (11)$$

where $\nu_0 = 1{\cdot}1446/V^2$, V being the coefficient of variation for d/σ. Log c was then obtained from the relation (9).* Table 6 gives values of ν, λ and log c for the representative sample sizes $n = 5, 10, 20, 30, 50$ and columns 5, 6 and 7 in that table give the difference in β_2 values for a variable λ, difference in β_1 values for the fixed λ and the difference in β_2 values for the fixed λ, respectively. Table 1 gives values of ν and log c for this fixed $\lambda = 1{\cdot}3219$ for $n = 5(1)\,20, 25, 30, 40, 50$. We need not repeat the arguments and the conclusions in this case which are similar to those in the case of δ^2/σ^2 presented in § 4 above. A comparison of the β_2 differences in Tables 3 and 6 suggests that the approximation to the d/σ distribution may

Table 6. *Approximation to* d/σ *by* $(\chi^2_\nu/c)^\alpha$

n	ν	λ	log c	β_2 difference	$\lambda = 1{\cdot}3219$	
					β_1 difference	β_2 difference
5	6·16	1·3101	0·7047	+0·0445	−0·0066	+0·0301
10	13·02	1·3189	1·0375	+0·0157	−0·0008	+0·0144
20	26·8	1·3221	1·3549	+0·0070	−0·0001	+0·0077
30	40·5	1·3242	1·5354	−0·0015	+0·0002	+0·0065
50	67·9	1·3261	1·7608	+0·0058	+0·0001	+0·0028

be closer than that for δ^2/σ^2. In his 1953 paper (p. 120) Kamat gave a table of percentage points of the distribution of d/σ obtained by using a Pearson-curve approximation, having the correct first four moments. We have compared these results with those obtained from the $(\chi^2_\nu/c)^\alpha$ approximation now recommended (with $\lambda = 1{\cdot}3219$) at $n = 5$, 10 and 50 at both upper and lower 0·5, 1·0, 2·5 and 5·0 % points and have found a most satisfactory agreement. The two series of figures were usually identical to the two decimal places given and never differed by more than a unit in the second decimal place. We do not, of course, know the true distribution of d/σ, but general experience suggests that the 4-moment Pearson curve representation is likely to be good.† We think, therefore, that our present approximation in terms of a power of χ^2 is likely to be entirely adequate for our purpose.

7. STATISTICS BASED ON SECOND SUCCESSIVE DIFFERENCES: δ_2^2/σ^2 AND d_2/σ

The same method, taking the moments given by Kamat (1954, pp. 4, 7), was used to approximate to the distributions of δ_2^2/σ^2 and d_2/σ. Here also by trial we found that a fixed

* Unlike δ^2/σ^2 the formula (11) was found to be useful for all $n \geqslant 5$ and linear interpolation in Brownlee's tables for V^2 was not necessary.

† In the single case $n = 3$, where the distribution of d/σ is far from normal, having $\beta_1 = 1{\cdot}0356$, $\beta_2 = 4{\cdot}2962$, Kamat (1953*b*, p. 121) was able to derive the true distribution and found that the 4-moment fit corresponded very closely with the true curve.

$\lambda = 0{\cdot}7353$ is most suitable for δ_2^2/σ^2 for $n \geqslant 7$ and a fixed $\lambda = 1{\cdot}2195$ is adequate for d_2/σ^2 for $n \geqslant 5$.*

In the case of δ_2^2/σ^2 for the fixed $\lambda = 0{\cdot}7353$, ν was obtained from

$$\nu = \nu_0 + 0{\cdot}1296 - 0{\cdot}2406\nu_0^{-1} + \ldots, \tag{12}$$

where $\nu_0 = 3{\cdot}6992/V^2$, V being the coefficient of variation of δ^2 (this formula is useful for $n \geqslant 10$; for $n < 10$, the exact value of ν, is obtained by linear interpolation as explained in § 4). For d_2/σ with $\lambda = 1{\cdot}2195$ kept fixed, ν was calculated from

$$\nu = \nu_0 + 0{\cdot}0324 - 0{\cdot}0358\nu_0^{-1} + \ldots, \tag{13}$$

Table 7. *β_1 and β_2 differences for the $(\chi_\nu^2/c)^\alpha$ approximation to the distributions of δ_2^2/σ^2 and d_2/σ using a fixed λ*

n	δ_2^2/σ^2, $\lambda = 0{\cdot}7353$		d_2/σ, $\lambda = 1{\cdot}2195$	
	β_1 difference	β_2 difference	β_1 difference	β_2 difference
5	+0·0749	+0·4517	−0·020	+0·088
7	−0·0008	+0·1400	−0·008	+0·045
10	−0·0008	+0·0782	−0·001	+0·031
15	+0·0005	+0·0482	+0·001	+0·022
20	+0·0014	+0·0354	+0·002	+0·016
25	+0·0009	+0·0273	+0·002	+0·013
30	+0·0009	+0·0222	+0·001	+0·011
40	+0·0007	+0·0162	+0·002	+0·006
50	+0·0006	+0·0129	+0·002	+0·007

where $\nu_0 = 1{\cdot}3448/V^2$ and V is the coefficient of variation of d_2. In both cases $\log c$ was then calculated from the equation (9).

Values of ν and $\log c$ for δ_2^2/σ^2 and d_2/σ for sample sizes $n = 5, 7, 10, 15, 20, 25, 30, 40, 50$ have been already given in Table 1. Discrepancies in the β_1 and β_2 values for fixed values of λ are given in Table 7. We have not discussed here these approximations to the distributions of δ_2^2/σ^2 and d_2/σ to the extent we have done above in the case of δ^2/σ^2 and d/σ, since they are in less frequent use than the latter.

In conclusion we wish to thank Prof. E. S. Pearson, who has drawn the attention of one of us to Cadwell's approximation and suggested improvements in the drafting of the paper.

* For $n = 5$ the difference in β_2 was considerably larger for a fixed λ than for a variable λ for δ_2^2/σ^2, but not so for d_2/σ. This is borne out by the following values of β_2 differences for variable λ compared with those given in Table 7.

	n	ν	λ	β_2 difference
δ_2^2/σ^2	5	3·418	0·7463	+0·2939
d_2/σ	5	4·71	1·1998	+0·126

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QUEUEING WITH BALKING

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1. Introduction

In dealing with problems of queueing, several writers (Kolmogoroff, 1932; Erlang*; Kendall, 1951, 1953; Lindley, 1952; Takács, 1955) have discussed the situation where queue stability is obtained by assuming that the demand for service does not overload the service mechanism. Thus λ, the average number of arrivals per unit time, is assumed to be less than μ, the average number of departures per unit time, so that their ratio, ρ, is less than unity. Kawata (1955), on the other hand, has shown that queue stability can also be obtained by assuming that, although arrivals occur more frequently than departures, some arrivals choose not to join the queue. In theory this case can be included in the original one, simply by supposing λ to be computed only from the values provided by those who actually join the queue. However, if the decision to join or not depends on some random variables, it is sensible to inquire into the relationship between these variables and those which characterize the queue, such as queue length and waiting time.

The factors which influence the decision of a person to join a queue or not may be considered under two general headings: (*a*) those relating to the importance of being served, and (*b*) those relating to the obstacle which the queue presents, namely the waiting time which he must experience. It is obvious that the waiting time cannot be found without a knowledge of the service times of all those in the queue; we shall, therefore, make the simplifying assumption that the individual measures the obstacle presented by the queue by its length when he arrives, which will be denoted by $k(t)$.

The factors included in (*a*) may be much more complicated, and may produce an opinion ranging from absolute urgency, so that a queue of arbitrary length will be joined, to absolute indifference, so that no non-zero queue will be joined. It will be assumed that these factors have been weighed by the individual before he arrives, and have produced in his mind an integer K, which is the greatest queue length that he will tolerate. If then he observes $k(t) \leqslant K$ on arrival, he joins the queue; but if $k > K$, he goes away and does not return. We assume that the values K chosen by the various individual arrivals may be regarded as being random samples from a certain distribution—the balking distribution. We shall sometimes allow K to be infinite with non-zero probability, so that a finite proportion of the arrivals will join any queue. In this paper we assume that if a person joins a queue, he must remain for service. In another paper we will permit him to test sequentially whether to stay or go, while he waits.

While the principal interest here is $\rho \geqslant 1$, it should be noted that the classic queueing cases are included, and that the appropriate results can be obtained by letting $K \to \infty$ when $\rho < 1$.

These results are (implicitly) special cases of a general set-up considered by Kendall & Reuter (1957). We derive them by assuming equilibrium, and investigate them numerically for various balking distributions.

* See Brockmeyer *et al.* (1948).

2. DIFFERENTIAL EQUATIONS

Suppose both arrivals and departures occur as events of homogeneous Poisson processes, of density λ, μ, respectively. Let

$$P(x,t) = \Pr(k \leqslant x \text{ at time } t), \quad F(x) = \Pr(K \leqslant x),$$
$$Q(x,t) = 1-P(x,t), \quad G(x) = 1-F(x),$$
$$p(x,t) = P(x,t)-P(x-1,t), \quad f(x) = F(x)-F(x-1).$$

Note that in general $p(0,t) = P(0,t) \neq 0$; this is the probability that no queue exists at time t. Also $f(0) = F(0) \neq 0$; this is the probability that an individual is absolutely queue-resistant. The distribution of K will be called the balking distribution.

Given a queue of length x, then the probability that an arrival joins it

$$= \Pr(\text{his balking value} \geqslant x)$$
$$= G(x-1).$$

The build-up of differential equations, following Feller, is

$$p(x,t+\Delta t) = [1-(\lambda+\mu)\Delta t]\,p(x,t)+\mu p(x+1,t)\Delta t + \lambda p(x-1,t)\,G(x-2)\Delta t+\lambda p(x,t)\,F(x-1)\Delta t+O(\Delta t)^2,$$

where the four contributions on the right-hand side are from

(*a*) no arrivals or leavers in Δt,
(*b*) one leaver in Δt,
(*c*) one arrival in Δt who joins,
(*d*) one arrival in Δt who balks,

respectively.

Take $p(x,t)$ from both sides, divide by Δt, and take the limit

$$(\partial/\partial t)\,p(x,t) = -(\lambda+\mu)\,p(x,t)+\mu p(x+1,t)+\lambda p(x-1,t)\,G(x-2)+\lambda p(x,t)\,F(x-1),$$

where, if $x = 0$, we delete μ in the first term; $F(-1) = 0 = F(-2)$. Add over x, giving

$$(\partial/\partial t)\,P(x,t) = \mu p(x+1,t)-\lambda p(x,t)\,G(x-1). \tag{1}$$

Writing $\lambda G(x) = \lambda_x$, (1) can be reduced to a form similar to the equation for a birth-and-death process given by Feller (1950), but differing both with respect to one subscript and with respect to the initial equation, namely

$$(\partial/\partial t)\,P(x,t) = \lambda_{x-1}P(x-1,t)-(\mu+\lambda_{x-1})\,P(x,t)+\mu P(x+1,t).$$

If, however, each person must join the queue, so that $F(x) = 0$ for all x, then (1) becomes a special case of the birth-and-death equation.

Using a method suggested by Koopman (1953) we can give a method for computing solutions of (1) in cases where they exist. Let $\phi(x,s)$ be the Laplace transformation of $p(x,t)$. Transforming, (1) becomes

$$-\lambda G(x-2)\,\phi(x-1,s)+(x+\lambda G(x-1)+\mu)\,\phi(x,s)-\mu\phi(x+1,s) = \delta_x,$$

where $\phi(-2,s) = \phi(-1,s) = 0$, $\delta_x = 1$ for $x = 0$ and 0 otherwise, thus assuming the queue empty at $t = 0$. Writing

$$\psi(x,s) = \phi(x,s)/\phi(x-1,s)$$

and using the equations corresponding to $x = 1, 2, \ldots$, it is seen that $\psi(1, s)$ may be written as a continued fraction

$$\psi(1,s) = \frac{\lambda}{B_1 -} \frac{\lambda\mu G(0)}{B_2 -} \frac{\lambda\mu G(1)}{B_3 -} \cdots.$$

Substituting this value back into the equation for $x = 0$, we find $\phi(0, s)$, and hence each $\phi(x, s)$. Values of $p(x, t)$ may then be found by the numerical inversion of the Laplace transform.

3. Equilibrium distributions

If equilibrium distributions of queue length exist as $t \to \infty$, they will be denoted by suppressing the letter t, and can be found from (1) by setting the left side equal to zero

$$p(1) = \rho p(0), \quad p(x+1) = \rho G(x-1)\, p(x) \quad (x = 1, 2, \ldots). \tag{2}$$

Defining $$c_0 = c_1 = 1, \quad c_x = \prod_0^{x-2} G(i) \quad (x = 2, 3, \ldots)$$

this can be written $$p(x) = \rho^x c_x p(0). \tag{3}$$

Summing, $$1 = p(0) \sum_0^\infty \rho^x c_x.$$

Kawata (1955) and Kendall & Reuter (1957) have shown that the convergence of the series $\sum_0^\infty \rho^x c_x$ is necessary and sufficient for equilibrium to be attained, replacing the classical condition $\rho < 1$ (which is contained); the condition is that $p(0) > 0$.

Thus, the tails of the balking distribution are proportional to the ratio of ordinates of the queue length distribution, and when ρ is known, either may be computed from the other. Some examples will be given in § 5.

From (2), we have (if $p(x), p(x+1) \neq 0$)

$$f(x) = G(x-1) - G(x) = \frac{p(0)}{p(1)} \left(\frac{p(x+1)}{p(x)} - \frac{p(x+2)}{p(x+1)} \right), \tag{4}$$

so we must have $$p^2(x+1) \geqslant p(x)\, p(x+2). \tag{5}$$

If $p(n) \neq 0$ but $p(n+1) = 0$, then from (2) we have $G(n-1) = 0$ and so $p(x) = 0$ for all $x > n$. This is the case where there is complete balking for queues of length n or more. Any finite distribution of queue-length satisfying (5) is attainable in this way; the condition for equilibrium will be satisfied for all ρ.

Any infinite distribution of queue-length (satisfying (5)) will correspond to a balking distribution (from (4)); there will be a positive probability that K is infinite (i.e. $G(\infty) \neq 0$) unless

$$\lim_{x \to \infty} \frac{p(x+1)}{p(x)} = 0. \tag{6}$$

If (6) is not satisfied, there is an upper limit to ρ for equilibrium to be attained. We have $c_x \sim A(G(\infty))^x$, so $\Sigma \rho^x c_x$ converges only if

$$\rho < \frac{1}{G(\infty)} = \lim_{x \to \infty} \frac{p(x)}{p(x+1)}.$$

The reverse problem, of finding $p(x)$ when $f(x)$ is given can be solved, at least numerically, in every case. However, there are some algebraic difficulties in the three steps of an analytical solution: (*a*) expressing $\rho G(x)$ in a convenient form, (*b*) evaluation of c_x, and (*c*) summing the series $\Sigma \rho^x c_x$.

4. GENERATING FUNCTIONS

Let
$$\eta(s,t) = \sum_0^\infty s^x p(x,t) = (1-s)\sum_0^\infty s^x P(x,t),$$
$$\xi(s,t) = \sum_0^\infty s^x p(x+1,t)\,F(x).$$

Summing (1) after multiplication by s^x, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\eta(s,t)}{1-s}\right) = -\lambda p(0,t) + \left(\frac{\mu}{s}-\lambda\right)\{\eta(s,t)-p(0,t)\} + \lambda s\xi(s,t).$$

Letting $t\to\infty$ with the usual change of notation, we have (assuming equilibrium)

$$\eta(s) - \rho s\eta(s) - p(0) + \rho s^2\xi(s) = 0. \tag{7}$$

With $s=1$,
$$\eta(0) = p(0) = 1-\rho+\rho\xi(1)$$

and since $0<p(0)\leqslant 1$,
$$\frac{\rho-1}{\rho} < \xi(1) \leqslant 1.$$

It will be noted that $\xi(1)$ is just the (asymptotic) probability that an individual balks. Recalling that ρ was calculated for all arrivals, whether they joined or not, it can now be seen that an effective value of the traffic intensity, computed only from those who join the queue, say ρ', can be written $\rho' = \rho - \rho\xi(1)$.

Writing now $p_\rho(x)$, $\eta_\rho(x)$, m_ρ (mean queue length), etc., to denote the dependence on ρ, we have from (3)

$$\eta_\rho(s) = p_\rho(0)\sum_0^\infty s^x\rho^x c_x = \sum_0^\infty (\rho s)^x c_x \Big/ \sum_0^\infty \rho^x c_x = p_\rho(0)/p_{\rho s}(0), \tag{8}$$

and using (7)
$$\xi_\rho(s) = \frac{p_\rho(0)}{\rho s^2}\left\{1 - \frac{1-\rho s}{p_{\rho s}(0)}\right\}. \tag{9}$$

Thus the queue length and balking distributions are uniquely determined once $p_\rho(0)$ is known as a function of ρ.

We have from (3)

$$\rho\frac{\partial}{\partial\rho}\log p_\rho(x) = \rho\frac{\partial}{\partial\rho}\left\{\log c_x + x\log\rho - \log\sum_0^\infty \rho^x c_x\right\} = x - m_\rho \tag{10}$$

which essentially determines $p_\rho(x)$ (and in particular $p_\rho(0)$) when m_ρ is known (as a function of ρ). Also, the variance of the queue length,

$$v_\rho = \sum_0^\infty p_\rho(x)\,(x-m_\rho)^2 = \sum_0^\infty (x-m_\rho)\rho\frac{\partial}{\partial\rho}p_\rho(x) = \rho\frac{\partial}{\partial\rho}m_\rho \tag{11}$$

(since $\Sigma p_\rho(x) = 1$).

$\eta_\rho(s)$ and $\xi_\rho(s)$ each satisfy a simple partial differential equation; from (8) we have

$$\log\eta_\rho(s) = \log\eta_\rho(0) + g_1(\rho s)$$

whence
$$\left(\rho\frac{\partial}{\partial\rho} - s\frac{\partial}{\partial s}\right)\log\eta_\rho(s) = \rho\frac{\partial}{\partial\rho}\log\eta_\rho(0) = -m_\rho.$$

Also from (9)
$$\log\xi_\rho(s) = \log\eta_\rho(0) - \log\rho s^2 + g_2(\rho s),$$
$$\left(\rho\frac{\partial}{\partial\rho} - s\frac{\partial}{\partial s}\right)\log\xi_\rho(s) = -m_\rho - 1 + 2 = 1 - m_\rho,$$

where $g_1(\ )$ and $g_2(\ )$ are used to denote 'a function of' and 'another function of' in the above equations.

Differentiating (7) and setting $s = 1$ gives

$$m_\rho = \frac{\rho}{1-\rho}\{1-\xi'_\rho(1)-2\xi_\rho(1)\}. \tag{12}$$

Differentiating (7) twice and setting $s = 1$ gives

$$v_\rho = \frac{\rho}{(1-\rho)^2}\{1-\rho[\xi'^2_\rho(1)+4\xi^2_\rho(1)+4\xi_\rho(1)\,\xi'_\rho(1)]-(1-\rho)[\xi''_\rho(1)+5\xi'_\rho(1)+4\xi_\rho(1)]\}. \tag{13}$$

5. Some examples

(i) *Binomial queue*

Condition (5) is satisfied by the finite distribution

$$p(x) = {}^nC_x\pi^x(1-\pi)^{n-x} \quad (x = 0, 1, \ldots, n),$$

where $0 < \pi < 1$. We find

$$\rho = \frac{n\pi}{1-\pi}, \quad 0<\rho<\infty; \qquad m_\rho = \frac{n\rho}{n+\rho} \to n \quad \text{as} \quad \rho\to\infty.$$

The corresponding balking distribution is given by

$$G(x) = \begin{cases} \dfrac{n-x-1}{n(x+2)} & 0 \leqslant x \leqslant n-1, \\ 0 & n \leqslant x, \end{cases} \qquad f(x) = \begin{cases} \dfrac{n+1}{n}\dfrac{1}{(x+1)(x+2)} & 0 \leqslant x \leqslant n-1 \\ 0 & n \leqslant x \end{cases}$$

and is independent of π. As $\rho\to\infty$, $\pi\to 1$ and the queue is almost always of length just n. The relations (8) to (11) may be verified immediately:

$$p_\rho(0) = \left(\frac{n}{n+\rho}\right)^n, \quad \eta_\rho(s) = \left(\frac{n+\rho s}{n+\rho}\right)^n, \quad v_\rho = \rho\left(\frac{n}{n+\rho}\right)^2.$$

(ii) *Negative binomial queue*

An example of an infinite queue-length distribution is

$$p(x) = {}^{N+x-1}C_{N-1}\chi^x(1+\chi)^{-N-x} \quad (x = 0, 1, 2, \ldots),$$

where $\chi > 0$ and $N > 1$. We have

$$\lim_{x\to\infty}\frac{p(x+1)}{p(x)} = G(\infty) = \frac{1}{N},$$

$$\rho = \frac{N\chi}{1+\chi}, \quad 0<\rho<N; \qquad m_\rho = \frac{N\rho}{N-\rho} \to \infty \quad \text{as} \quad \rho\to N.$$

The balking distribution is

$$G(x) = \frac{N+x+1}{N(x+2)}, \quad f(x) = \frac{N-1}{N}\frac{1}{(x+1)(x+2)} \quad (x = 0, 1, 2, \ldots),$$

with a probability $1/N$ at $x = \infty$. The balking distribution is independent of χ:

$$p_\rho(0) = \left(\frac{N-\rho}{N}\right)^N, \quad \eta_\rho(s) = \left(\frac{N-\rho}{N-\rho s}\right)^N, \quad v_\rho = \rho\left(\frac{N}{N-\rho}\right)^2.$$

(iii) *Poisson queue*

Each of the queue distributions in (i) and (ii) approaches the Poisson form as $n, N \to \infty$:

$$p(x) = \frac{\rho^x e^{-\rho}}{x!} \quad (x = 0, 1, 2, \ldots).$$

We now have $G(\infty) = 0$, so $0 < \rho < \infty$; $m_\rho = \rho = v_\rho$. The balking distribution becomes

$$G(x) = \frac{1}{x+2}; \quad f(x) = \frac{1}{(x+1)(x+2)} \quad (x = 0, 1, 2, \ldots)$$

which may be regarded as a discrete analogue of the Cauchy distribution

$$p_\rho(0) = e^{-\rho}, \quad \eta_\rho(s) = e^{-\rho+\rho s}.$$

(iv) *Type III ordinates*

Another possible infinite queue-length distribution is

$$p(x) = A(x+a)^\nu e^{-\lambda x} \quad (x = 0, 1, 2, \ldots),$$

where $a, \nu, \lambda > 0$; A is a complicated function of a, ν and λ. Here

$$G(\infty) = \left(\frac{a}{a+1}\right)^\nu, \quad \rho = \left(\frac{a+1}{a}\right)^\nu e^{-\lambda}, \quad \text{so} \quad 0 < \rho < \left(\frac{a+1}{a}\right)^\nu.$$

The balking distribution is

$$G(x) = \left[\frac{a(a+x+2)}{(a+1)(a+x+1)}\right]^\nu \quad (x = 0, 1, 2, \ldots)$$

which is independent of λ. For ρ near $(a+1)^\nu/a^\nu$, we have approximately

$$m_\rho \simeq \frac{\nu+1}{\log\left[\frac{1}{\rho}\left(\frac{a+1}{a}\right)^\nu\right]} - a, \quad v_\rho \simeq \frac{(m_\rho+a)^2}{\nu+1}.$$

If we let $\nu \to 0$ we obtain the classic case where every arrival must join the queue

$$p(x) = \frac{e^{-\lambda x}}{1-e^{-\lambda}}, \quad G(\infty) = 1, \quad \rho = e^{-\lambda}, \quad 0 < \rho < 1.$$

(v) *Normal ordinates*

A very simple result is obtained when we assume that the queue-length distribution is

$$p(x) = A \exp -\frac{(x-m)^2}{2v} \quad (x = 0, 1, 2, \ldots),$$

where m and v are very nearly the mean and variance of the distribution if v and m^2/v are 'large', say both > 9. We find

$$G(\infty) = 0, \quad \rho = \exp(m - \tfrac{1}{2})/v, \quad 0 < \rho < \infty.$$

The balking distribution is Pascal (geometric)

$$G(x) = \exp -(x+1)/v, \quad f(x) = (1-\lambda)\lambda^x \quad (x = 0, 1, 2, \ldots), \tag{14}$$

where $\lambda = \exp v^{-1}$, and is independent of m. Denoting the mean of t[illegible] distribution by $M = \lambda/(1-\lambda)$, we have

$$m_\rho \simeq m = \tfrac{1}{2} + v \log \rho, \quad v_\rho \simeq v = \left[\log\left(\frac{M+1}{M}\right)\right]^{-1}. \tag{15}$$

The relation $m_\rho = \text{const.} + v_\rho \log \rho$ cannot hold exactly for any distribution whatever, as from (11) it implies first $v_\rho = c = \text{const.}$ and then $m_\rho < 0$ for $\rho < \exp -(c/v)$.

(vi) *Deterministic balking*

One case in which these calculations are easy is the following: each person possesses exactly the same degree of queue resistance, i.e. the balking distribution is deterministic, $f(x) = 1$ for $x = K$ and zero otherwise. Although rather trivial, this case will be important in an application to be mentioned subsequently. We have

$$G(x) = \begin{cases} 1 & 0 \leqslant x \leqslant K-1, \\ 0 & K \leqslant x, \end{cases} \qquad c_x = \begin{cases} 1 & 0 \leqslant x \leqslant K+1, \\ 0 & K+2 \leqslant x, \end{cases}$$

so
$$p_\rho(0) = \frac{1-\rho}{1-\rho^{K+2}}; \qquad p(x) = \begin{cases} \dfrac{1-\rho}{1-\rho^{K+2}}\rho^x & 0 \leqslant x \leqslant K+1, \\ 0 & K+2 \leqslant x. \end{cases}$$

From (8) and (9), or direct from the definitions, we have

$$\xi_\rho(s) = \frac{1-\rho}{1-\rho^{K+2}}\rho^{K+1}s^K, \quad \eta_\rho(s) = \frac{1-\rho}{1-\rho^{K+2}}\frac{1-(\rho s)^{K+2}}{1-\rho s}.$$

Hence, from (12) and (13), or from (10) and (11), we have

$$m_\rho = \frac{\rho}{1-\rho} - (K+2)\frac{\rho^{K+2}}{1-\rho^{K+2}},$$

$$v_\rho = \frac{\rho}{(1-\rho)^2} - (K+2)^2\frac{\rho^{K+2}}{(1-\rho^{K+2})^2}.$$

In the trivial situation where each individual insists on immediate service, $K = 0$, and

$$m_\rho = \frac{\rho}{1+\rho}, \quad v_\rho = \frac{\rho}{(1+\rho)^2}.$$

In the classic situation, in which each individual must join the queue, $K = \infty$, and

$$m_\rho = \frac{\rho}{1-\rho}, \quad v_\rho = \frac{\rho}{(1-\rho)^2}.$$

Each of (i) to (v) above has led to a balking distribution which is unimodal with mode at $x = 0$. It seems very difficult to obtain explicit results for a balking distribution with mode not zero; numerical calculations were carried out for $\rho = 2, 3, 4, 5, 6, 7$, for each of the following balking distributions:

(vii) Poisson with mean 10.

(viii) Ordinates of a lognormal distribution $f(y)$ where $5(\log y - 1)$ is a unit normal variable.

(ix) Ordinates of a χ^2 distribution with 10 degrees of freedom.

(x) The Pascal distribution (14) with $\lambda = 10/11$.

Table 1 shows the means and variances of the queue-length distributions obtained, together with those of the 'input' balking distributions. Also, in the last column, are given the approximate values for the Pascal distribution (x) obtained from (15). It will be seen that the approximation is very close except when $\rho = 2$; here m^2/v is only 5·5.

Table 1. *Mean and variance of queue length for various balking distributions*

	Poisson	Lognormal	χ^2	Pascal	Formulae (15)
Input					
Mean	10	10·57	9·498	10	10
Variance	10	19·5384	19·23	110	110
$\rho = 2$					
Mean	10·320206	10·212608	9·475843	7·823407	7·772564
Variance	4·908937	6·506500	6·110196	9·726074	10·49205
$\rho = 3$					
Mean	11·941052	12·578738	11·633753	12·026812	12·026641
Variance	3·351061	5·382238	4·725268	10·467543	10·49205
$\rho = 4$					
Mean	12·826379	14·078101	12·914084	15·041116	15·045024
Variance	2·844219	5·074961	4·248282	10·490947	10·49205
$\rho = 5$					
Mean	13·430370	15·192921	13·830718	17·363218	17·386325
Variance	2·582158	4·929776	4·032884	10·463681	10·49205
$\rho = 6$					
Mean	13·885518	16·086535	14·547050	19·284075	19·299236
Variance	2·416736	4·858583	3·839912	10·533106	10·49205
$\rho = 7$					
Mean	14·248864	16·831880	15·130198	20·919959	20·916585
Variance	2·301017	4·813526	3·728680	10·485757	10·49205

6. SOME GENERALIZATIONS

An interesting application of queueing with balking is furnished by the problem of a sequence of transporting mechanisms which move discrete units of cargo. In the terminology developed by the Department of Engineering of the University of California at Los Angeles (1953), each transporting agency constitutes a 'link', and the place where two such mechanisms transfer their loads is a 'node'. If there is room at a node for the storage of S items, the number in storage at time t may be regarded as a queue of length $0, 1, \ldots, S$. If the link setting down items carries A_1 units at a time, and the link picking up items carries A_2 at a time, then we must assume $S \geqslant A_1 + A_2 - 1$, to prevent the process stopping altogether. The time required for the journeys of the links (in which we absorb the time required for pick up or set down) constitute the inter-arrival and inter-departure times of the queue at the node between them. If the storage $k(t)$ at the node is such that the arriving link cannot set down its cargo, i.e. if $S - k(t) < A_1$, the link goes back and arrives again after another inter-arrival time. If the storage when removing link arrives satisfies $k(t) < A_2$, so that a full load is not available, it also goes back and forth until the inequality is reversed.

Thus, considering only two links separated by one node, the following generalizations of the probabilistic queueing model discussed in our introduction are suggested: (*a*) bulk

arrivals and departures, (*b*) balking distributions associated with both arrivals and departures, (*c*) a finite number of states possible. Also, the cargo-handling case requires deterministic balking; we shall begin with stochastic balking, and then specialize for cargo-handling.

Let K_1 and K_2 be two integers such that when A_1 items arrive, they are added to the queue when $k(t) \leqslant K_1$ and not otherwise, and when the removal mechanism arrives, it takes away A_2 items if $k(t) \geqslant K_2$ and not otherwise. Let $F_j(x) = \Pr(K_j \leqslant x) = 1 - G_j(x)$.

Using the same argument that was employed in § 2,

$$\begin{aligned} p(x, t+\Delta t) = {} & (1-\lambda\Delta t-\mu\Delta t)\, p(x,t) \\ & + \lambda\Delta t(1-\mu\Delta t)\,[p(x,t)\,F_1(x-1) + p(x-A_1,t)\,G_1(x-A_1-1)] \\ & + \mu\Delta t(1-\lambda\Delta t)\,[p(x,t)\,G_2(x) + p(x+A_2,t)\,F_2(x+A_2)] + o(\Delta t), \end{aligned}$$

where departures as well as arrivals are now partitioned into two cases. Also, we have special cases not only for $x = 0$, but for x less than A_1 or A_2. However, this can be dealt with by conventions regarding negative arguments. We put

$$\begin{aligned} p(x) &= 0 \quad \text{for} \quad x<0 \quad \text{or} \quad x>S, \\ F_1(x) &= 0 \quad \text{for} \quad x<0, \\ F_2(x) &= 0 \quad \text{for} \quad x\leqslant 0. \end{aligned}$$

Passing to the limit, and letting $t \to \infty$ (assuming equilibrium) we obtain

$$0 = \rho G_1(x-A_1-1)\,p(x-A_1) + \{\rho G_1(x-1) + F_2(x)\}\,p(x) - F_2(x+A_2)\,p(x+A_2) \qquad (x = 0, 1, \ldots, S).$$

These $S+1$ equations are linearly dependent; there is the further relation $\Sigma p(x) = 1$. The system of equations may be written $\mathbf{Bp} = \mathbf{a}$, where

$$\mathbf{p} = \{p(0), p(1), \ldots, p(S)\}, \quad \mathbf{a} = \{0, 0, \ldots, 1\}$$

and $\mathbf{B}$ is an $(S+1)\times(S+1)$ matrix having zeros everywhere except in the last row (where all the elements are unity), the principal diagonal, the A_2th super-diagonal, and the A_1th subdiagonal.

If $A_1 = A_2 = 1$, these equations can be solved exactly as in § 3; only a redefinition of c_x is needed

$$c_x = \prod_0^{x-2} G_1(j) \Big/ \prod_1^{x} G_2(j).$$

In case of arbitrary A_1 and A_2, but with deterministic balking defined by

$$K_1 = S - A_1, \quad K_2 = A_2$$

(which are the values suggested by the cargo-handling problem), the matrix $\mathbf{B}$ simplifies as follows:

(*a*) the elements in the A_1th subdiagonal become $-\rho$,

(*b*) the elements in the A_2th super-diagonal become -1,

(*c*) the principal diagonal consists of $-A_2$ elements of value ρ, followed by

$$S - A_1 - A_2 + 1 \geqslant 0$$

elements of value $1+\rho$ and, finally, A_1 elements of value 1.

I have evaluated this determinant only in special cases, but offer the following conjectures about the polynomial $B(\rho; A_1, A_2, S)$:

(i) it vanishes unless A_1 and A_2 are relatively prime;

(ii) if A_1 and A_2 are relatively prime, it consists of $S-A_1-A_2+3$ terms, beginning with a term of degree $S-A_1+1$ and ending with a term of degree A_2-1;

(iii) $B(\rho; m, n, S) = \rho^S B(\rho^{-1}; n, m, S)$;

(iv) the first and last coefficients are A_1 and A_2, respectively;

(v) $B(\rho; 1, 1, S) = \sum_{j=0}^{S} \rho^j$.

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TESTING FOR SERIAL CORRELATION IN SYSTEMS OF SIMULTANEOUS REGRESSION EQUATIONS

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1. Introduction

Much attention has been given in recent years to the problem of fitting economic models and this has led to the study of systems of simultaneous regression equations of the form

$$Ay = Bx + \epsilon, \tag{1}$$

where y is a vector of jointly dependent variables, x is a vector of predetermined or independent variables, ϵ is a vector of errors, and A and B are matrices of unknown parameters. It is usual to assume that A is square and non-singular, that the ϵ's are random variables with zero means and constant variance matrix, and that the x's are either constants or random variables distributed independently of the ϵ's. Given the model (1) we usually require to estimate the elements of A and B from a sample of n observations of y and x and to make confidence statements about the estimates.

In some formulations certain of the x's coincide with lagged values of the y's. The theory becomes much more complicated in such cases, and we shall not consider them except to point out that the results obtained later in the paper, which are exact for the model specified above, may be expected to hold approximately for models containing lagged dependent variables.

The specification (1) is a natural generalization of the ordinary linear regression model and calls for similar methods of analysis. However, a number of new difficulties present themselves, among which is the *identification* problem, that is the question whether the parameter values can be inferred from a complete knowledge of the distribution of the observations. It turns out that this is possible under reasonable assumptions only when certain restrictions are obeyed by the elements of A and B. Roughly speaking, when the restrictions are just sufficient to enable the parameters of a particular equation to be estimated the equation is said to be *just-identified*, and when the restrictions are more than sufficient the equation is said to be *over-identified*. More detailed explanations are given by Girshick & Haavelmo (1947) and Koopmans and others (1950).

When an equation is identified its parameters can be consistently estimated by the reduced-form method or the limited-information method, a treatment of which has been given by Anderson & Rubin (1949, 1950). Although some writers take the phrases as equivalent, in the present paper we shall use the phrase reduced-form method to refer only to estimation in the just-identified case while limited-information method will be taken to refer only to estimation in the over-identified case. These methods have the advantage that the parameters of the particular equation are estimated independently of the remaining parameters of the system. This is convenient since in many investigations we are not interested in the estimation of all the parameters of the system but only of those in a specific equation. The methods involve a possible loss of efficiency since the restrictions on the elements of A and B other than those in the equation under study are ignored in the

estimation procedure. With this reservation, when the errors are normally distributed and satisfy reasonable assumptions, the estimators so obtained are maximum likelihood estimators.

A method for the simultaneous estimation of all the parameters of the system which makes use of all the information available about the restrictions on the elements of A and B has been developed by Koopmans and others (1950), but it requires extremely heavy computations and has other disadvantages which make it unlikely that the method will be found useful in practice. Consequently, it will be assumed in this paper that the equations under study have been fitted by reduced-form or limited-information methods.

These methods are extensions of the method of least squares to the simultaneous equations case, and like least squares depend for their validity on the assumption that the errors are serially uncorrelated. If this assumption does not hold, the estimators are not efficient and confidence regions calculated without taking the serial correlation into account may be highly misleading. Tests of serial correlation should therefore form part of any analysis in which the observations consist of time series. A small-sample test for single-equation models has been given by Durbin & Watson (1950, 1951). The purpose of the present paper is to extend this test to cover simultaneous-equation models in which the parameters have been fitted by the methods referred to.

It is, of course, possible to transform (1) by multiplying through by A^{-1} to give the system

$$y = A^{-1}Bx + A^{-1}\epsilon, \tag{2}$$

in which each equation contains only a single dependent variable. If the original errors ϵ are serially uncorrelated, so are the transformed errors $A^{-1}\epsilon$ and each equation may be fitted separately by least squares. Consequently, the test appropriate to single-equation models may be applied to each equation separately and in this way it appears that the hypothesis that all the ϵ's are serially uncorrelated may be investigated. In view of this, it may be asked why any special consideration need be given to simultaneous-equation models since on the face of it the problem can be dealt with by existing methods.

In the first place there is no method available for combining the results from the separate tests into an overall test for all the errors in the model. Secondly, even if an overall test could be constructed it would not be suitable for cases in which our attention is focused on a specific equation of the model. What is required is a test which has high power against alternatives affecting estimators of the parameters in that equation. As will be seen below, for a sample of n observations these estimators depend on a vector of n variates which are functions of the errors in the equations of (1). The estimation procedure depends on the partitioning of this vector into a component lying in a space of r dimensions, called the estimation space, r being the number of independent variables in the model, and a component in a space of $n-r$ dimensions, called the residual space. On the usual assumptions of normality of the errors, the efficiency of the estimates and the validity of confidence statements about them depends on the property that these components are randomly directed in their respective spaces. The effect of departure from serial independence is to make the distribution of directions non-random. Now the estimation component depends on the unknown parameters that are being estimated whereas the direction of the residual component is independent of unknown parameters. Thus, it seems reasonable to base the test of serial dependence on the distribution of directions of the residual component. The statistic used in the tests described below measures the direction of this component and is known to have high power against

Markoff alternatives. Details of its power properties may be found in the paper by Durbin & Watson (1950).

Reference to a procedure attributed to H. Rubin for testing serial correlation in simultaneous-equation models is made on pp. 70 and 73 of the book by Klein (1950). Details are not given, but the method is based on the lag correlations of the complete set of residuals from all equations of the system, and so is quite different from the test described in this paper. Presumably Rubin's is a large-sample test, whereas the test described below is valid for small samples.

2. Just-identified equations

Suppose that the model (1) has $p+1$ equations and that we wish to estimate the parameters in the first of these equations, which we assume is just-identified. Suppose also that the vector x has $k+p$ elements ($k \geqslant 0$). It is convenient to write (1) in the transposed form

$$y'A' = x'B' + \epsilon'. \tag{3}$$

The model for n observations is then

$$YA' = XB' + E, \tag{4}$$

where Y, X are $n \times (p+1)$ and $n \times (k+p)$ matrices of observations and E is an $n \times (p+1)$ matrix of errors. The rows of E are assumed to be independent vectors from the same multivariate normal population with zero variate means. We consider the usual case in which the restrictions on the elements of $A = [\alpha_{ij}]$ and $B = [\beta_{ij}]$ which ensure identification are $\alpha_{11} = 1$ and

$$\beta_{k+1.1} = \ldots = \beta_{k+2.1} = \ldots = \beta_{k+p.1} = 0.$$

Representing column vectors by small letters and matrices by capital letters (4) can then be written in the form

$$[y_1 \mid Y_2]\begin{bmatrix} 1 & \alpha_1' \\ \alpha & A_2 \end{bmatrix} = [X_1 \mid X_2^*]\begin{bmatrix} \beta & B_1^* \\ O_p & B_2 \end{bmatrix} + [\epsilon_1 \mid E_2], \tag{5}$$

where α and β are $p \times 1$ and $k \times 1$ vectors whose elements are to be estimated and O_p is a $p \times 1$ vector of zeros. The values of the elements of the vector α_1 and the matrices A_2, B_1^* and B_2 do not concern us except that B_2 is required to have rank p. ϵ_1 is the $n \times 1$ vector of errors in the first equation and E_2 is the $n \times p$ matrix of errors in the remaining equations of the system. In this notation the n observations of the first equation are written as

$$y_1 + Y_2\alpha = X_1\beta + \epsilon_1. \tag{6}$$

Let X_2 denote the part of X_2^* orthogonal to X_1, i.e. $X_2 = X_2^* - X_1(X_1'X_1)^{-1}X_1'X_2^*$. (5) can then be rewritten as

$$[y_1 \mid Y_2]\begin{bmatrix} 1 & \alpha_1' \\ \alpha & A_2 \end{bmatrix} = [X_1 \mid X_2]\begin{bmatrix} \beta & B_1 \\ O_p & B_2 \end{bmatrix} + [\Sigma_1 \mid E_2]. \tag{7}$$

For the purpose of developing the test of serial correlation we may without loss of generality assume that the columns of X_1 and X_2 are normalized and orthogonal, i.e. $X_1'X_1$ and $X_2'X_2$ are unit matrices. This assumption is permissible since the test will be based on the residuals from the fitted equation and these are unaffected by the assumption.

The reduced-form estimator a of α is obtained by equating to zero the regression of $y_1 + Y_2a$ on X_2, i.e. by setting $X_2'(y_1 + Y_2a) = O_p$, which gives $a = -(X_2Y_2)^{-1}X_2'y_1$. The estimator of b is given by the regression of $y_1 + Y_2a$ on X_1, i.e. $b = X_1'(y_1 + Y_2a)$. The set of n residuals from the fitted equation (6) is then given by $z = y_1 + Y_2a - X_1b$.

As in the single-equation case (Durbin & Watson, 1950, 1951) the test for serial correlation is performed by calculating the statistic

$$\left.\begin{aligned} d &= \frac{\sum_{i=2}^{n}(z_i - z_{i-1})^2}{\sum_{i=1}^{n} z_i^2} \\ &= \frac{z'Dz}{z'z}, \end{aligned}\right\} \tag{8}$$

where D is the matrix

$$\begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 1 \end{bmatrix}.$$

We shall show that in the usual case in which one of the columns of X_1 is the vector of 1's $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, the significance of d may be tested by entering Durbin & Watson's (1951) tables with $k' = k+p-1$.

Post-multiplying (4) by $(A')^{-1}$ we have

$$Y = XC + F,$$

or, in the form corresponding to (7),

$$[y_1 \vdots Y_2] = [X_1 \vdots X_2]C + (f_1 \vdots F_2]. \tag{9}$$

Substituting in $z = y_1 + Y_2 a - X_1 b,$

we have $z = X_1 c_1 + X_2 c_2 + f,$

where c_1 and c_2 are given vectors depending on a, b and $f = f_1 + F_2 a$.

Let K be an $n-k-p \times n$ matrix such that $H = \begin{bmatrix} X_1' \\ X_2' \\ K \end{bmatrix}$ is an orthogonal matrix, and let $\xi = Hz$. Then

$$\begin{aligned} \xi &= \begin{bmatrix} X_1' \\ X_2' \\ K \end{bmatrix} [X_1 c_1 + X_2 c_2 + f] \\ &= \begin{bmatrix} O_{k+p} \\ u \end{bmatrix}, \end{aligned}$$

where u is the $n-k-p$ vector Kf.

Let a set of independent normal variates (vectors) with zero means and constant variance (variance matrix) be called $INZC$ variates (vectors). Since the rows of E are $INZC$ vectors, so are the rows of F, and since H is orthogonal so are the rows of HF.

Consider the distribution of u conditional on the first $k+p$ rows of HF being held fixed. The estimating equation for a is $X_2'(y_1 + Y_2 a) = O_p$, which reduces to $c_2 + X_2'(f_1 + F_2 a) = O_p$. Thus the value of a depends only on the first $k+p$ rows of HF, and is therefore conditionally fixed. Consequently u behaves conditionally as if a were a fixed quantity. Now u depends only on a and the last $n-k-p$ rows of HF. Since the rows of HF are $INZC$ vectors of whose

elements u is a linear combination, it follows that u is conditionally a vector of $n-k-p$ *INZC* variates.

Since $\xi = Hz$, we have $z = H'\xi$. Substituting in (8) we have

$$d = \frac{\xi' HDH'\xi}{\xi'\xi}$$

$$= \frac{u'Lu}{u'u},$$

where L is an $n-k-p \times n-k-p$ matrix.

Let N be the orthogonal matrix diagonalizing L, i.e.,

$$N'LN = \begin{bmatrix} \nu_1 & & & \\ & \nu_2 & & \\ & & \ddots & \\ & & & \nu_{n-k-p} \end{bmatrix}$$

the blank spaces representing zeros. Let $\eta = N'u$, i.e. $u = N\eta$. Then

$$d = \frac{\sum_{i=1}^{n-k-p} \nu_i \eta_i^2}{\sum_{i=1}^{n-k-p} \eta_i^2}, \tag{10}$$

where $\eta_1, \ldots, \eta_{n-k-p}$ are conditionally *INZC* variates. Denote their common (conditional) variance by σ^2 and let $\zeta_i = \eta_i/\sigma$ $(i = 1, \ldots, n-k-p)$. Then $\zeta_1, \ldots, \zeta_{n-k-p}$ are independent $N(0, 1)$ variates. Substituting in (10) we have

$$d = \frac{\sum_{i=1}^{n-k-p} \nu_i \zeta_i^2}{\sum_{i=1}^{n-k-p} \zeta_i^2}. \tag{11}$$

Now the *INZC* property of $\xi_1, \ldots, \xi_{n-k-p}$ depended on the restrictions imposed on the first p rows of HF, whereas $\zeta_1, \ldots, \zeta_{n-k-p}$ are independent $N(0, 1)$ independently of these restrictions. We may therefore relax the restrictions and the distribution of d defined by (11) will hold unconditionally.

d is now in the form considered for the single-equation case by Durbin & Watson (1950, especially pp. 411–13, and 1951). It is shown in these papers that $d_L \leqslant d \leqslant d_U$, where

$$d_L = \frac{\sum_{i=1}^{n-k-p} \lambda_i \zeta_i^2}{\sum_{i=1}^{n-k-p} \zeta_i^2} \quad \text{and} \quad d_U = \frac{\sum_{i=1}^{n-k-p} \lambda_{k+p+i} \zeta_i^2}{\sum_{i=1}^{n-k-p} \zeta_i^2},$$

$\lambda_1 < \lambda_2 < \ldots < \lambda_n$ being the latent roots of D. Significance points of d_L and d_U were tabulated by Durbin & Watson and in their notation $k+p-1 = k'$. It may be noted that in each case k' is the number of constants fitted in addition to the mean. For a test against positive serial correlation an observed d less than the tabulated value of d_L is declared significant, while an observed value greater than the tabulated d_U is declared non-significant. For intermediate values of d the test is inconclusive; however, an approximate procedure to deal with this contingency was given by Durbin & Watson and this applies equally to the present case. The method requires the calculation of the first two moments of d in terms of the matrix $X - \bar{X}$ of

deviations from the means of the $k+p-1$ independent variables (other than that corresponding to the mean) in the model. For tests against negative serial correlation and two-sided tests the procedures given by Durbin & Watson may be followed.

The results of this section may be summarized in the following rule:

When a just-identified equation has been fitted by the reduced-form method a test of serial correlation may be performed by calculating

$$d = \frac{\sum_{i=2}^{n} (z_i - z_{i-1})^2}{\sum_{i=1}^{n} z_i^2},$$

where $z_1, \ldots, z_n$ is the set of observed residuals, and entering the tables d_L and d_U (Durbin & Watson, 1951) with k' equal to the number of constants fitted in addition to the mean.

3. OVER-IDENTIFIED EQUATIONS

When the number of independent variables with zero coefficients in the first equation is greater than p the equation is over-identified. The model for n observations is, as before,

$$YA' = XB' + E,$$

where Y, E are $n \times p+1$ matrices but X is now $n \times k+q$ with $q > p$. The restrictions on B which give rise to over-identification are $\beta_{k+1.1} = \beta_{k+2.1} = \ldots = \beta_{k+q.1} = 0$.

Equation (4) can again be written in the form

$$[y_1 \vdots Y_2] \left[\begin{array}{c|c} 1 & \alpha' \\ \hline \alpha & A_2 \end{array}\right] = [X_1 \vdots X_2] \left[\begin{array}{c|c} \beta & B_1 \\ \hline O_q & B_2 \end{array}\right] + [\epsilon_1 \vdots E_2], \tag{12}$$

where $X = [X_1 \vdots X_2^*]$ and $X_2 = X_2^* - X_1(X_1'X)^{-1}X_1'X_2^*$. If the columns of X_1 and X_2 are not already normalized and orthogonal we may transform to new independent variables with this property without affecting the values of the residuals on which the serial correlation test is to be based.

The limited-information estimator of α is the vector a such that in the Euclidean representation of the $[n-k]$-dimensional space orthogonal to the $[k]$ space spanned by the columns of X_1, the angle between the projection on the $[n-k]$ space and the vector $y_1 + Y_2 a$ and the $[q]$ space spanned by the columns of X_2 is a maximum. It may be noted that in the just-identified case, for which $q = p$, this angle is a right angle; when $q > p$, the angle will in general be less than a right angle. The limited-information estimator b of β is given by the regression of $y_1 + y_2 a$ on X_1, i.e. $b = X_1'(y_1 + Y_2 a)$.

Let $\xi = H(y_1 + Y_2 a - X_1 b)$, where K is such that $H = \begin{bmatrix} X_1' \\ X_2' \\ K \end{bmatrix}$ is an orthogonal matrix. Then $\xi = \begin{bmatrix} O_k \\ v \\ u \end{bmatrix}$, where $v = X_2'(y_1 + Y_2 a)$ is the projection of the residual vector z on the X_2 space and $u = K(f_1 + F_2 a)$ where, as in § 2, the rows of $F = [f_1, F_2] = E(A')^{-1}$ are $INZC$ vectors.

We choose a to mimimize $(v'v/u'u)$. As a function of the errors v depends only on the elements of $X_2'F$ and $u'u$ depends only on the elements of the $p+1 \times p+1$ matrix $F'K'KF$. Thus, a depends only on the elements of $X_2'F$ and $F'K'KF$.

Now the rows of KF are $INZC$ vectors independent of the other rows of HF. Hence, their joint density depends only on the elements of the matrix of sums of squares and products, namely $F'K'KF$. Consequently, all samples for which $F'K'KF$ is held constant are equally

likely. Now the elements of $F'K'KF$ fix the lengths and relative inclinations of the vectors constituting the columns of KF. Regarding these vectors as a set of axes, if $F'K'KF$ is fixed the configuration of the axes is fixed, but all orientations of the axes in the $n-k-q$ space are equally likely. Consequently, any vector whose orientation is fixed relative to the axes is randomly directed in the $n-k-q$ space.

Consider now the distribution of u conditional on the $X_2'F$ and $F'K'KF$ held fixed. Then a is a fixed vector, so that $u = K(f_1+F_2a)$ is conditionally a fixed linear combination of the columns of KF. It follows that u is randomly directed in the $n-k-q$ space, i.e. the joint distribution of the elements of u is spherically symmetric.

Let b_2 denote the vector of least-squares coefficients of the regression of y_1+Y_2a on X_2, i.e. $b_2 = X_2'(y_1+Y_2a)$, and let $z^* = y_1+Y_2a-X_1b-X_2b_2$. Then z^* denotes the set of residuals from the regression of y_1+Y_2a on X_1 and X_2. The test for serial correlation is performed by calculating the statistic

$$d = \frac{\sum_{i=2}^{n}(z_i^*-z_{i-1}^*)^2}{\sum_{i=1}^{n} z_i^{*2}}.$$

Let $\xi^* = Hz^*$. Then $\xi^* = \begin{bmatrix} O_{k+q} \\ u \end{bmatrix}$. As in (10),

$$d = \frac{\sum_{i=1}^{n-k-q} \nu_i \eta_i^2}{\sum_{i=1}^{n-k-q} \eta_i^2},$$

where $\eta = N'u$, N' being an orthogonal matrix.

Consider the conditional distribution of d for fixed X_2F and $F'K'KF$. Since N' is orthogonal and points u are equi-probable on the sphere $u'u =$ constant, points η are equi-probable on the sphere $\eta'\eta =$ constant. Put $\eta_i^* = \eta_i/(\eta'\eta)^{\frac{1}{2}}$. Then $d = \sum_{i=1}^{n-k-q} \nu_i \eta_i^{*2}$, where points η^* are equi-probable on the unit sphere. Since $\eta_1^*, \ldots, \eta_{n-k-q}^*$ are independent of $X_2'F$ and $F'K'KF$, the unconditional distribution of d when $X_2'F$ and $F'K'KF$ are allowed to vary is identical with the conditional distribution when these quantities are held fixed.

Let w be a χ^2 variate with $n-k-q$ degrees of freedom independent of $\eta_1^*, \ldots, \eta_{n-k-q}^*$, and let $\zeta_i = w^{\frac{1}{2}}\eta_i^*$ $(i = 1, \ldots, n-k-q)$. Then $\zeta_1, \ldots, \zeta_{n-k-q}$ are independent $N(0, 1)$ variates and

$$d = \frac{\sum_{i=1}^{n-k-q} \nu_i \zeta_i^2}{\sum_{i=1}^{n-k-q} \zeta_i^2}.$$

This is in the same form as the expression (11) obtained for the just-identified case. It follows that an observed d may be tested by entering Durbin & Watson's (1951) tables with $k' = k+q-1$. When the test is inconclusive the same approximate procedure may be used as in the single-equation case, based on the $n \times k+q-1$ matrix $X-\bar{X}$ of deviations from the sample means of the $k+q-1$ independent variables other than the mean.

We have assumed in developing the theory of the test that the columns X_1, X_2 are normalized and orthogonal, but it is not necessary to transform the variables to satisfy this condition in order to apply the test in practice. In fact the extra calculations required for the

test additional to those necessary for calculating the limited-information estimates, are not as onerous as the treatment given above might suggest.

The usual computing procedure for the limited-information method (Klein, 1953, § 4·3, p. 169) requires the calculation of the matrix

$$W = Y'Y - Y'X(X'X)^{-1}X'Y,$$

where Y, X represents matrices of deviations from means of observations of dependent variables in the equation and independent variables in the model (whether in the equation or not) respectively. Normally the matrix product $Y'X(X'X)^{-1}X'Y$ is calculated by means of a single Doolittle procedure as described by Klein. When a serial-correlation test is envisaged it will be more convenient to calculate $(X'X)^{-1}X'Y$ first and obtain the product by multiplication by $Y'X$. Apart from this the vectors a, b of limited-information estimates are calculated in the way described by Klein.

Let a^* denote the $p+1 \times 1$ vector $\begin{bmatrix} 1 \\ a \end{bmatrix}$. Then the set of residuals z^* on which the test is based is the set of residuals from the least-squares regression of Ya^* on X, i.e.

$$z^* = [I - X(X'X)^{-1}X']\, Ya^*.$$

It is worth noting that the sum of squares $\sum_{i=1}^{n} z^{*2}$ is equal to $a^{*\prime}Wa^*$.

We may summarize the results of this section in the following rule: When the equation $y_1 + Y_2\alpha = X_1\beta + \epsilon$ has been fitted by the limited-information method, a test of serial correlation may be performed by calculating

$$d = \frac{\sum_{i=2}^{n} (z_i^* - z_{i-1}^*)^2}{\sum_{i=1}^{n} z_i^{*2}},$$

where $z_1^*, \ldots, z_n^*$ is the set of residuals from the multiple regression of $y_1 + Y_2 a$ (a being the limited-information estimate of α) on all the independent variables of the system, whether or not they occur in the fitted equation with non-zero coefficients, and referring to Durbin & Watson's (1951) Table 1, with $k' = k + q - 1$.

In practice it is unlikely that there will be much difference between the value of d calculated from the residuals z and that calculated from the residuals z^*. In fact the author conjectures that the bounds test applies to the former value as well as to the latter, although he has been unable to prove this. Thus, if it is wished to avoid the extra labour of calculating the residuals z^*, it seems that a good approximate test could be obtained by calculating d from the residuals z and referring to Durbin & Watson's table with $k' = k + q - 1$.

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HETEROGENEOUS ERROR VARIANCES IN SPLIT-PLOT EXPERIMENTS

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SUMMARY. In a split-plot experiment, the common assumption is that the same error variance applies to all subplot treatments. This paper is concerned with tests of significance for the departure from equality of the variances for different subplot treatments, and with the estimation of the ratio of a pair of such variances. The methods used may be regarded as extensions of those described by Morgan (1939) and Pitman (1939) in connexion with the problem of comparing the variances of two correlated and normally distributed measurements. In particular, Pitman's method is extended to provide confidence limits for the variance ratio.

Suppose that visual acuities were measured, for right eye and left eye separately, on a sample of men, and that the possibility of a difference between σ_r^2 and σ_l^2, the variances for the two eyes, was to be investigated. If the distribution of the original measurements is bivariate normal, then so is that of their sum and their difference. Morgan and Pitman noted that the correlation coefficient in the latter distribution depends upon the difference $(\sigma_r^2-\sigma_l^2)$, being zero if and only if $\sigma_r^2=\sigma_l^2$. Consequently, they proposed to test the hypothesis of equal variance by the standard significance test for a sample correlation coefficient. In an agricultural experiment of split-plot design the variance for a subplot may depend upon the treatment it receives. In particular this is liable to happen if one subplot treatment increases the yields by a considerable amount. The simple procedure for testing the equality of two variances outlined above would allow for the correlation between the subplots of each whole-plot, but is no longer applicable, because of treatment and block classifications. The more general methods of this paper enable tests of significance and confidence limits to be calculated.

An example of a simple split-plot experiment is given, with a discussion of the standard errors for the various treatment comparisons.

1. Introduction

The simplest type of split-plot experiment consists of c blocks of a whole-plots, with each whole-plot split into b subplots. Denote the a whole-plot treatments by $A_1, \ldots, A_a$ and the b subplot treatments by $B_1, \ldots, B_b$. The usual model on which the analysis is based takes

$$y_{ijk} = \mu+\delta_k+\alpha_i+\beta_j+\gamma_{ij}+\epsilon_{ijk} \quad \begin{cases}(i = 1, \ldots, a),\\ (j = 1, \ldots, b),\\ (k = 1, \ldots, c),\end{cases}$$

with

$$\sum_{k=1}^{c} \delta_k = \sum_{i=1}^{a} \alpha_i = \sum_{=1}^{b} \beta_j = 0;$$

$$\sum_{i=1}^{a} \gamma_{ij} = 0 \quad \text{all } j,$$

and

$$\sum_{j=1}^{b} \gamma_{ij} = 0 \quad \text{all } i;$$

where y_{ijk} is the observation in the kth block on the treatment combination A_iB_j, μ is the general mean, δ_k the block effect, α_i the effect of treatment A_i, β_j the effect of treatment B_j and γ_{ij} the interaction of treatments A_i and B_j.

The usual assumptions made about the errors are that each ϵ_{ijk} is the sum of a whole-plot error, with zero mean and variance σ_w^2, and a subplot error, with zero mean and variance σ_s^2, and further that the whole-plot and subplot errors are uncorrelated. We shall examine the more general model in which the ϵ_{ijk}'s are independently distributed about zero, except

for correlations between the ϵ's for the several subplots of each main plot, and in which the ϵ's for each subplot treatment may have a different variance. Thus we take

$$E(\epsilon_{ijk}^2) = \sigma_j^2,$$

$$E(\epsilon_{ijk}\epsilon_{ij'k}) = \rho_{jj'}\sigma_j\sigma_{j'}.$$

The more usual assumptions mentioned above would have

$$\sigma_j^2 = \sigma_w^2 + \sigma_\epsilon^2,$$

and

$$\rho_{jj'}\sigma_j\sigma_{j'} = \sigma_w^2 \quad \text{for all } j, j'.$$

This paper is concerned, in the main, with the particular case $b = 2$. Except when the contrary is explicitly stated, it is hereafter assumed that each main plot has only two subplots; of course, if $b > 2$ and a particular pair of subplot treatments is of interest, an analysis as for $b = 2$ can be performed for these subplots alone.

2. A test of the hypothesis $\sigma_1^2 = \sigma_2^2$

Define S and D by

$$S = y_{i1k} + y_{i2k},$$

$$D = y_{i1k} - y_{i2k}.$$

Further, let E_{SS}, E_{SD} and E_{DD} denote the error sums of squares and sums of products in the analysis of covariance of S and D shown in Table 1. The breakdown of the sums of squares for S and for D is the usual one for whole-plot totals and differences except that the $a(c-1)$ degrees of freedom for the subplot error have been split into $(c-1)$ for the interaction of treatments B and blocks and a further $(a-1)(c-1)$.

Table 1. *Analysis of covariance of S and D*

Components for analysis of S	d.f.	Sums of squares and products			d.f.	Components for analysis of D
		S^2	SD	D^2		
Adjustment for general mean	1	—	—	—	1	B
A	$a-1$	—	—	—	$a-1$	AB
Blocks	$c-1$	—	—	—	$c-1$	Subplot error
Whole-plot error	$(a-1)(c-1)$	E_{SS}	E_{SD}	E_{DD}	$(a-1)(c-1)$	
Total	ac				ac	Total

The covariance of S and D, in subplot units, is $\frac{1}{2}(\sigma_1^2 - \sigma_2^2)$. Therefore the F test for regression of S on D (or D on S) provides a test of the hypothesis $\sigma_1^2 = \sigma_2^2$. Under the null hypothesis $\sigma_1^2 = \sigma_2^2$, and the usual normality assumptions,

$$F = \frac{mE_{SD}^2}{E_{SS}E_{DD} - E_{SD}^2}$$

has a F distribution with 1 and m degrees of freedom where

$$m = (a-1)(c-1) - 1.$$

3. Confidence limits for σ_1^2/σ_2^2

Confidence limits for σ_1^2/σ_2^2 can be obtained by an extension of the method due to Pitman. Instead of the covariance analysis of S and D, consider the similar covariance analysis of

$$S' = \frac{y_{i1k}}{\sigma_1} + \frac{y_{i2k}}{\sigma_2},$$

and

$$D' = \frac{y_{i1k}}{\sigma_1} - \frac{y_{i2k}}{\sigma_2}.$$

Let the error sums of squares and products be denoted by E'_{SS}, E'_{SD} and E'_{DD}. S' and D' are uncorrelated and therefore the variance ratio for testing the regression coefficient of S' on D' against zero has a F distribution with 1 and m degrees of freedom, where m is again $ac-a-c$. Thus

$$\frac{\sqrt{m}\,E'_{SD}}{\sqrt{(E'_{SS}E'_{DD}-E'^2_{SD})}} = \sqrt{F} = t \qquad (1)$$

has a t distribution with m degrees of freedom.

Now

$$S' = \frac{S+D}{2\sigma_1} + \frac{S-D}{2\sigma_2},$$

and therefore

$$E'_{SS} = \frac{E_{SS}+E_{DD}+2E_{SD}}{4\sigma_1^2} + \frac{E_{SS}+E_{DD}-2E_{SD}}{4\sigma_2^2} + \frac{E_{SS}-E_{DD}}{2\sigma_1\sigma_2} = \frac{E_1}{4\sigma_1^2} + \frac{E_2}{4\sigma_2^2} + \frac{r\sqrt{(E_1E_2)}}{2\sigma_1\sigma_2},$$

where $\frac{1}{2}E_1$ and $\frac{1}{2}E_2$ are the error sums of squares and $\frac{1}{2}r\sqrt{(E_1E_2)}$ is the error sum of products in the covariance analysis of y_{i1k} and y_{i2k}.

[Of course E_1, E_2 and r could be obtained directly from a covariance analysis of y_{i1k} and y_{i2k}, but equality of σ_1^2 and σ_2^2 could not be tested by regarding E_1/E_2 as an ordinary variance ratio because of the correlation between the two subplots in each main plot. When $b = 2$ it seems more convenient to extend the usual analyses of variance of S and D into the covariance analysis of Table 1. On the other hand, if $b > 2$, a test of equality of two particular σ_j^2, and confidence limits for their ratio, may perhaps be obtained more readily by forming the analysis of covariance of the appropriate y_{ijk}, $y_{ij'k}$ and taking E_j, $E_{j'}$ and r from this.]

We similarly have

$$E'_{DD} = \frac{E_1}{4\sigma_1^2} + \frac{E_2}{4\sigma_2^2} - \frac{r\sqrt{(E_1E_2)}}{2\sigma_1\sigma_2},$$

and

$$E'_{SD} = \frac{E_1}{4\sigma_1^2} - \frac{E_2}{4\sigma_2^2}.$$

By substitution in (1)

$$t = \frac{\sqrt{m}\left[\frac{E_1}{\sigma_1^2} - \frac{E_2}{\sigma_2^2}\right]}{2\sqrt{\left(\frac{E_1E_2}{\sigma_1^2\sigma_2^2}(1-r^2)\right)}}.$$

If the α percentage point of the t distribution with m degrees of freedom is written τ then

$$\Pr(-\tau < t < +\tau) = 1-\alpha.$$

Rearrangement of the inequalities leads to the final result that

$$\Pr\left[\frac{E_1}{E_2}(K-\sqrt{(K^2-1)}) < \sigma_1^2/\sigma_2^2 < \frac{E_1}{E_2}(K+\sqrt{(K^2-1)})\right] = 1-\alpha,$$

where $$K = 1+\frac{2(1-r^2)\tau^2}{m};$$

$$E_1 = E_{SS}+E_{DD}+2E_{SD},$$

$$E_2 = E_{SS}+E_{DD}-2E_{SD},$$

and $$r^2 = \frac{(E_{SS}-E_{DD})^2}{(E_{SS}+E_{DD})^2-4E_{SD}^2};$$

$$m = (a-1)(c-1)-1,$$

and $\tau = \alpha$ percentage point of the t distribution with m degrees of freedom.

4. OTHER SPLIT-PLOT DESIGNS

If the whole-plot design is more complicated than simple randomized blocks (e.g. a Latin square), an exactly analogous covariance analysis to that given above will lead to the test of equality for the subplot variances and to the confidence limits for their ratio.

A further application is to Graeco-Latin squares in which the Greek and Latin letters represent two different sets of treatments, applied separately to the two halves of each whole-plot (Yates, 1937, § 16*i*). The errors associated with the two particular treatments occurring together in any whole-plot may be correlated and have different variances. Extending the analyses of variances of sums and differences (Yates, 1937) into a covariance analysis, we have, for a $p \times p$ square, Table 2.

Table 2. *Analysis of covariance of sums and differences for a Graeco-Latin square*

Components for analysis of S	d.f.	Sums of squares and products			d.f.	Components for analysis of D
		S^2	SD	D^2		
Adjustment for general mean	1	—	—	—	1	Latin *v.* Greek
Latin treatments	$p-1$	—	—	—	$p-1$	Latin treatments
Greek treatments	$p-1$	—	—	—	$p-1$	Greek treatments
Rows	$p-1$	—	—	—	$p-1$	
Columns	$p-1$	—	—	—	$p-1$	Subplot error
Whole-plot error	$(p-1)(p-3)$	E_{SS}	E_{SD}	E_{DD}	$(p-1)(p-3)$	
Total	p^2				p^2	Total

The extension of the covariance analysis to more than one square would follow the usual lines. The tests of significance and determination of confidence limits proceed exactly as above for the simpler type of split-plot design.

5. Example

In an experiment to compare the yields of six strains of cocksfoot (*Dactylis glomerata*), the strains were allocated at random to six whole-plots in each of four replicates. Each whole-plot was split into two for the comparison of high and low levels of fertilizer. Yields of grass from the first cut were about twice as much at the high level of fertilizer as at the low, and it was therefore suspected that the error variances for these two levels might be different.

The analysis of covariance is given in Table 3 and the test for the regression of S on D in Table 4. The value for F shows that σ_1^2 and σ_2^2 differ significantly at the 5 % level.

Table 3. *Analysis of covariance of S and D*

Components for analysis of S	d.f.	Sums of squares and products			d.f.	Components for analysis of D
		S^2	SD	D^2		
Adjustment for general mean	1	293,594	128,418	56,170	1	Fertilizer
Strains	5	30,784	−2,152	18,399	5	Interaction
Replicates	3	16,038	922	585	3	Subplot error
Whole-plot error	15	$37,618 = E_{SS}$	$19,817 = E_{SD}$	$39,330 = E_{DD}$	15	
Total	24	378,034	147,005	114,484	24	Total

Table 4. *Test for regression of S on D*

	d.f.	*S.S.*	*M.S.*	F
Regression of S on D	1	9,985	9,985	5·06
Residual	14	27,633	1,974	—
Whole-plot error	15	37,618	—	—

Now

$$E_1 = E_{SS} + E_{DD} + 2E_{SD} = 116{,}582,$$

$$E_2 = E_{SS} + E_{DD} - 2E_{SD} = 37{,}314,$$

$$E_1/E_2 = 3{\cdot}1244,$$

$$r^2 = \frac{(E_{SS} - E_{DD})^2}{E_1 E_2} = 0{\cdot}0007.$$

For 95 % confidence limits, τ is the 5 % point of the t distribution with $m = 14$ d.f. Thus $\tau = 2{\cdot}145$ giving $K = 1{\cdot}657$ and $\sqrt{(K^2 - 1)} = 1{\cdot}321$. Therefore 95 % confidence limits for σ_1^2/σ_2^2 are (1·05, 9·30).

Standard errors for various treatment comparisons

Let E_a and E_b be the whole-plot and subplot error mean squares based on 15 and 18 d.f., respectively. Then, if the error variances for the two subplot treatments are assumed equal, the standard errors for the various treatment comparisons are as follows:

Difference between two strain means $\sqrt{(\frac{1}{4}E_a)} = 25{\cdot}0$.

Difference between the two fertilizer means $\sqrt{(E_b/12)} = 13{\cdot}6$.

Difference between the two fertilizer means for a particular strain $\sqrt{(\frac{1}{2}E_b)} = 33{\cdot}3$.

Difference between two strain means at the same fertilizer level $\sqrt{[\frac{1}{4}(E_a+E_b)]} = 34{\cdot}4$.

If the subplot error variances are different, the last standard error needs to be reconsidered. The separate analysis of the two levels leads to exact t tests with 15 d.f. and the standard errors are, for the high level,

$$\frac{\sqrt{(E_{SS}+E_{DD}+2E_{SD})}}{2\sqrt{15}} = 44{\cdot}1,$$

and for the low level

$$\frac{\sqrt{(E_{SS}+E_{DD}-2E_{SD})}}{2\sqrt{15}} = 24{\cdot}9.$$

More precise estimates of the standard errors are

$$\frac{1}{2}\sqrt{\left(\frac{E_{SS}}{15}+\frac{E^*_{DD}+2E^*_{SD}}{18}\right)} = 41{\cdot}9 \quad \text{and} \quad \frac{1}{2}\sqrt{\left(\frac{E_{SS}}{15}+\frac{E^*_{DD}-2E^*_{SD}}{18}\right)} = 24{\cdot}6,$$

where E^*_{DD} and E^*_{SD} are the error sum of squares and sum of products for the differences based on all 18 d.f. The cost of including the extra 3 d.f. is that exact t tests are no longer possible.

I am indebted to Mr G. J. F. Copeman of the North of Scotland College of Agriculture for permission to use part of the results of an experiment on strains of cocksfoot in the numerical example.

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A MAXIMUM-MINIMUM PROBLEM RELATED TO STATISTICAL DISTRIBUTIONS IN TWO DIMENSIONS

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1. Statement of problem

We consider the following problem, and some developments of it. In its finite form it may be stated thus. A mass is distributed among the hk cells of a rectangle which has been divided into h rows and k columns. The content of each row and of each column is known, but the contents of the elementary cells are unknown. What is the least (or greatest) possible content of any given group of cells?

In the infinite form we have a density distribution $f(x, y)$ over the rectangle $0 \leqslant x \leqslant a$, $0 \leqslant y \leqslant b$. The group of cells is replaced by a given area which we call Region 1 or $R1$, the remainder of the rectangle being $R2$. Since the maximum content of $R1$ can be found by finding the minimum content of $R2$ we confine ourselves to minima. We require m, where

$$m = \min \iint_{R1} f(x, y)\, dx\, dy \tag{1}$$

subject to

$$\left.\begin{aligned} f(x, y) &\geqslant 0, \\ \int_0^b \int_0^x f(x, y)\, dx\, dy &= A(x), \\ \int_0^y \int_0^a f(x, y)\, dx\, dy &= B(y), \end{aligned}\right\} \tag{2}$$

where $A(x)$ and $B(y)$ are given monotonic functions increasing from O to $A(a) = B(b) = N$, N being the total mass. It may also be necessary to assume that $A(x)$ and $B(y)$ possess finite derivatives at every point.

It is convenient, as in Fig. 1, to take the origin of co-ordinates at the top left-hand corner of the rectangle, and to take the x axis to the right and the y axis down. We shall use the $A(x)$ and $B(y)$ notation, even in the finite form of the problem, to denote the total content of the columns to the left of the point (x, y), and the rows above it, respectively.

The problem in its finite form may be solved by Linear Programming methods. It is a special case of the Transportation Problem discussed by Hitchcock (1941) and Vajda (1955) with transport charges either zero or unity. The solution is obtained here by a different method.

2. Solution

It is obvious that m, the minimum content of $R1$, cannot be less than the minimum content of a region entirely contained in $R1$, in particular, it cannot be less than that of a rectangle contained in $R1$. Consider the minimum contents of all such rectangles and find the greatest, say m', then

$$m \geqslant m'. \tag{3}$$

It will be proved that the equality sign holds.

The minimum content of a region equals the minimum content of the enclosed rectangle of greatest minimum content. In this connexion the word rectangle stands not only for a compact rectangle, but for any area which can be formed into a compact rectangle by a rearrangement of rows and columns. Such a rearrangement has no effect on the problem, at least in the finite form. The usefulness of this result lies in the fact that the minimum content of a rectangle can be immediately written down, and that for simple forms of $R1$ the rectangle of greatest minimum content can easily be discovered. It is not necessary to find even one of the distributions of mass which actually satisfy the minimum condition.

3. PROOFS

The proofs which follow are complete for the finite form of the problem, both for unrestricted and for integral values of the cell contents. The infinite form has been proved subject to a restriction which involves both the nature of the area $R1$ and the marginal distributions $A(x)$ and $B(y)$. Some developments of the problem are discussed: they include the minimum content of an area contained in $R1$ when the content of $R1$ is maintained at its minimum.

(i) *Minimum content of a rectangle*

Consider Fig. 1. Vertical lines are denoted by L, horizontal lines by M, and rectangular areas by the lines which bound them. Region 1, whose minimum is required, is the rectangle $L_0L_1M_0M_1$ in the top left-hand corner. Suppose that the cell at P in $R1$ is not empty and that a cell in Q in L_1LM_1M is also not empty. Then without violating the marginal conditions we can transfer some mass from P to P' provided we transfer an equal amount from Q to Q'. This move reduces the content of $R1$. The process can be continued until either $R1$ or L_1LM_1M becomes empty. If $R1$ empties, its minimum content is clearly zero. If L_1LM_1M empties, the content of $L_1LM_0M_1$ is $N-A(x)$ that of $L_0L_1M_1M$ is $N-B(y)$, and consequently the content of $R1$ is $A(x)+B(y)-N$. It is easily seen that this is the least possible content of $R1$; any smaller content requires a negative content in the diagonally opposite rectangle L_1LM_1M. The result may be summed up as follows. If

$$A(x)+B(y)-N>0 \tag{4a}$$

the minimum content is $A(x)+B(y)-N$, and the necessary and sufficient condition that the content should have this minimum value is that the diagonally opposite rectangle L_1LM_1M should be empty. If

$$A(x)+B(y)-N\leqslant 0 \tag{4b}$$

then the minimum content is zero.

(ii) *Finite problem, general $R1$, cell contents not restricted to integral values*

Any row or column which is completely empty may be disregarded without affecting the problem. It is convenient to assume that this has been done, so that no row or column is completely empty. It is convenient also to refer to the processes we employ in these proofs in an easily picturable way. Therefore, although it is essential to the present proof that we do not restrict ourselves to integral values of the cell contents, we visualize the problem as one of arranging on the rectangle an immensely large number of little balls each representing an equal almost infinitesimal mass. We shall refer to an element of $R1$ as a 1-space, of $R2$ as a 2-space. In like manner a ball in a 1-space is a 1-ball, in a 2-space a 2-ball. Although we use the language of the integral problem, referring to individual balls for example, we do not

assume any definite number of balls and may change the number in the course of the argument if necessary.

Suppose first that a solution of the minimum problem has been found. Certain spaces contain balls, certain spaces are empty. If another solution of the problem is possible, a superposition of the two solutions is also a solution (the number of balls is different of course in the three solutions, but as already explained this is irrelevant). The new solution has the property that any cell which contained a ball in either of the first two solutions must contain a ball in the new solution. By superposing all possible solutions we therefore obtain a solution in which any space which can contain a ball does contain one. This is the solution with which we conduct the argument.

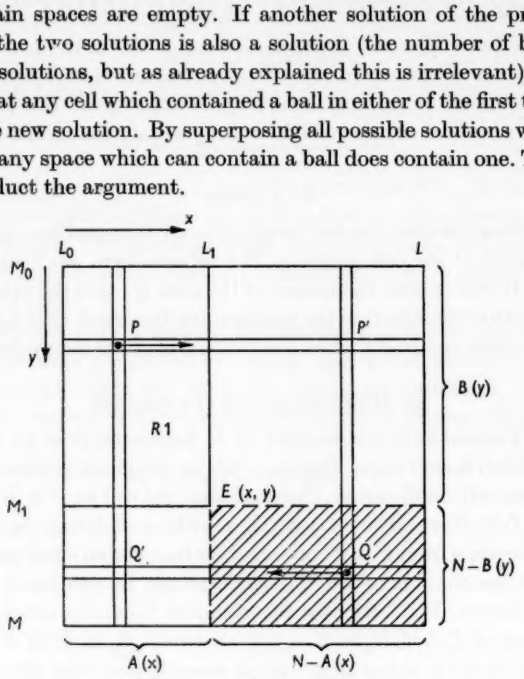

Fig. 1. Rectangular region 1.

Suppose now that a column contains a 1-ball, as for example at P in Fig. 1, where L_1 and M_1 are now to be ignored. Suppose also that the cell at Q' in this column is a 2-space. Then the row which contains Q' cannot contain a 1-ball. For suppose there were a 1-ball at Q, then we might move the ball at Q to Q', and the ball at P to the corresponding position P'. The totals in the rows and columns are unchanged so that the boundary conditions are still satisfied. But in the move from Q to Q' a 1-ball has become a 2-ball. The move from P to P' cannot reverse this change and may even increase it, therefore the minimum content of Region 1 has been reduced. Since this is impossible, there cannot have been a 1-ball in the row containing Q'.

We now carry out a rearrangement of the rows and columns. All rows which contain a 1-ball are collected at the top; all columns containing a 1-ball at the left-hand side. In Fig. 2 this rearrangement defines the lines L_1 and M_1, any row above M_1 and any column to the left of L_1 contains a 1-ball. Because of the nature of the solution with which we are working (every space filled which can be) there is no solution which has a 1-ball outside the rectangle $L_0L_1M_0M_1$. It is easily seen that $L_0L_1M_0M_1$ consists entirely of 1-space. If any elementary rectangle within it were a 2-space we should have the situation which in the previous paragraph was proved impossible: a 2-space whose intersecting row and column both contain a 1-ball.

To indicate in Fig. 2 that every row within a certain area contains a ball, we mark in the rows and put a dot, representing a ball, at the end of each row. The rectangle $L_0L_1M_0M_1$ has both rows and columns marked in this way because both rows and columns contain a 1-ball. To indicate in a similar way the presence of an elementary 1-space or 2-space we use the same convention but replace the dot by a small circle.

We next carry out a rearrangement of the rows below M_1 and the columns to the right of L_1. We collect at the bottom those rows which have a 2-space to the left of L_1, thus obtaining the dividing line M_2. Every row in $L_0L_1M_2M$ therefore contains a 2-space, as shown. In a similar way we obtain L_2, so that each column in $L_2LM_0M_1$ contains a 2-space. The regions $L_0L_1M_1M_2$ and $L_1L_2M_0M_1$ consist, like $L_0L_1M_0M_1$, entirely of 1-space since not to do so would contradict the principle by which L_2 and M_2 were defined. Since these regions are entirely 1-space, yet lie outside the region occupied by 1-balls, they must also be empty.

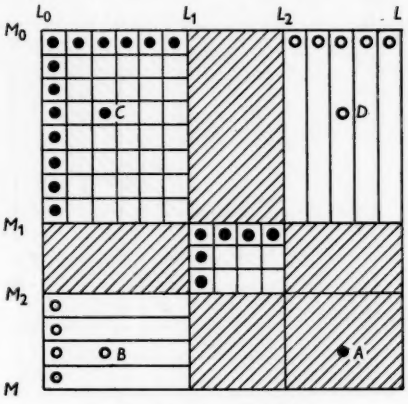

Fig. 2. Solution, finite form.

Consider now the rectangle L_1LM_2M, which will be proved empty. Suppose that it is not empty and that element A contains a 2-ball. Then since A lies below M_2, there is a 2-space at some space B in the same row as A and to the left of L_1. Again, since every column in $L_0L_1M_0M_1$ contains a 1-ball there must be one at some space C. If we move the ball from A to B and compensate by a move from C to D we satisfy the boundary conditions. But the arrival of a ball at D is impossible. If D is a 1-space we have a solution of the problem with a 1-ball outside the rectangle $L_0L_1M_0M_1$ within which all 1-balls must be found. If D is a 2-space we have reduced the minimum content of Region 1. Since both these results are impossible the rectangle L_1LM_2M must be empty. Similarly L_2LM_1M is empty. The empty regions are shaded diagonally in Fig. 2.

Consider now the rectangle $L_0L_1M_0M_2$. It consists entirely of 1-space. Its content is the minimum content of Region 1, since it contains all the 1-balls and nothing else. The diagonally opposite rectangle L_1LM_2M is empty and this implies that the content of $L_0L_1M_0M_2$ is at its minimum. Thus m, the minimum content of Region 1, equals the minimum content of $L_0L_1M_0M_2$. If there were another rectangle lying entirely within Region 1, whose minimum content m' exceeded that of $L_0L_1M_0M_2$, we should have

$$m' > m. \tag{5}$$

But since this rectangle lies within Region 1 it cannot have a minimum exceeding that of Region 1, hence

$$m' \leqslant m. \tag{6}$$

Since (5) and (6) are incompatible there is no such rectangle whose minimum m' exceeds m; in other words $L_0L_1M_0M_2$ is a rectangle of greatest minimum content in Region 1. We may prove exactly the same thing of $L_0L_2M_0M_1$.

We have therefore shown that the minimum content of Region 1 equals the minimum content of a rectangle which lies entirely within Region 1 and has the greatest minimum content of all such rectangles.

When there is a unique rectangle of greatest minimum content, L_1 and L_2 coincide and so do M_1 and M_2. Since we are assuming that there are no completely empty rows or columns it is easily seen that this is the only possible arrangement compatible with the existence of a unique rectangle.

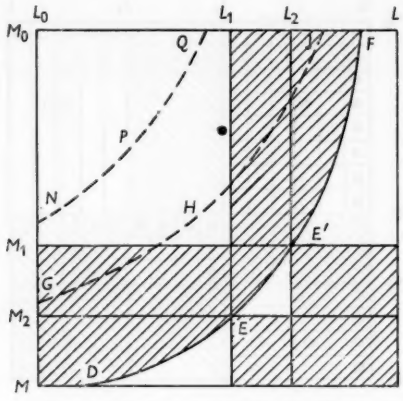

Fig. 3. Solution for monotonic curves.

It should perhaps be added that the proof is valid only when the minimum content of Region 1 is different from zero. But if the minimum content is zero, then the minimum content of any rectangle lying within Region 1 must also be zero. The theorem is therefore true generally.

When the division into Regions 1 and 2 is simple, it is usually easy to find the rectangle of greatest minimum content. Fig. 3 shows the simplest case: $R1$ is above a monotonic curve $DEE'F$. If we assume that $R1$ must be empty to the right of L_1, but that the column immediately to the left of L_1 contains a 1-ball, it follows that L_1LM_2M and all $R1$ below M_2 are empty spaces. Thus $L_0L_1M_0M_2$ is a rectangle of greatest minimum content, and it has a corner on the boundary curve. If $DEE'F$ is represented by $y = \phi(x)$ the content of such a rectangle is $A(x)+B(\phi(x))-N$, and m is the maximum of this function of x. For more complicated boundaries the rectangle may exist in separated pieces, but its content may be expressed as a function of x in a similar fashion.

(iii) *Finite problem, general R1, cell contents restricted to integral values*

The proof employs a process of rearrangement somewhat similar to that of the previous proof but more elaborate. We now have only N balls. If we have two solutions, the super-

position of these clearly does not produce a solution of the problem of N balls. It may not even be a solution of the problem of $2N$ balls allowing half integral values.

We assume in this case that we have found one integral solution of the problem. In Fig. 4 we collect as before into the regions M_0M_1 and L_0L_1 those rows and columns which contain a 1-ball. It is clear, as before, that $L_0L_1M_0M_1$ must consist entirely of 1-space, and that any balls outside this space at present must be 2-balls. Within $L_0L_1M_0M_1$ each row or column contains somewhere a 1-ball. This is indicated by the same convention as in Fig. 2.

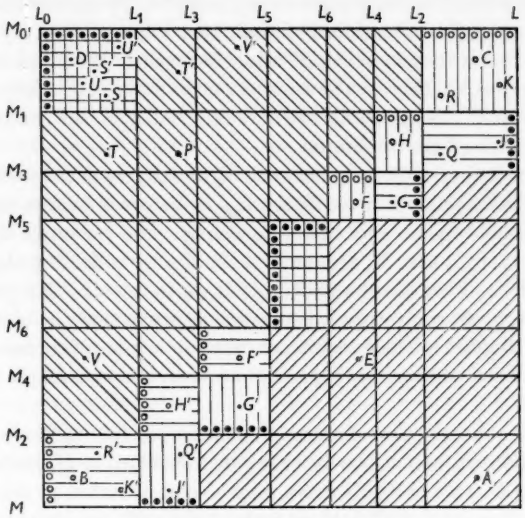

Fig. 4. Integral values only.

We now rearrange the remaining rows and columns. Consider the columns L_0L_1. All rows containing a 2-space within $L_0L_1M_1M$ are moved to the bottom of the diagram, thus forming the group M_2M. In a similar way we form the group L_2L. As indicated in the figure each row in $L_0L_1M_2M$ contains a 2-space, and each column in $L_2LM_0M_1$. The rectangles $L_0L_1M_1M_2$ and $L_1L_2M_0M_1$ consist entirely of 1-space and are therefore empty. We may now prove that L_2LM_2M is empty.

We show first that it is possible to rearrange the balls in $L_0L_1M_0M_1$ so as to obtain a ball in any desired cell. If the cell is empty the row and column which pass through it must each contain a 1-ball in some other cell. It is possible to remove these two balls and to replace them by one in the desired cell and one in the compensating position. All the cells concerned lie inside $L_0L_1M_0M_1$, and all are 1-spaces so that no conditions are broken by such a move. Thus we can obtain a ball in any desired position within $L_0L_1M_0M_1$.

Suppose that L_2LM_2M is not empty but contains a 2-ball at A. Then there must be 2-spaces at B and C in the same row and column as A. D is the space which completes the rectangle $ABCD$. We arrange to have a ball at D. The balls at A and D may now be replaced by balls at B and C, the net result of the move being a reduction in the 1-balls which are already at their minimum. This is impossible, so that L_2LM_2M must be empty.

Consider next the remaining space of the rows M_2M, i.e. $L_1L_2M_2M$. Columns which are empty in this rectangle we move to the right to join up with the empty rectangle L_2LM_2M.

The dividing line between empty and filled columns is L_3. The corresponding dividing line for $L_2LM_1M_2$ is M_3. We may now prove that $L_1L_3M_1M_3$ consists entirely of 1-space.

Suppose the cell at P were a 2-space. The row containing P must contain a 2-ball within $L_2LM_1M_3$; let this be at Q. This in turn is below a 2-space at say R, which again is level with a 1-ball at some point S in $L_0L_1M_0M_1$. We move the ball from Q to R and compensate from S to T. Since T is in 1-space the ball at T remains a 1-ball. In a similar way, using the moves Q' to R' and S' to T' we get another 1-ball at T'. But the balls at T and T' may be replaced by a 2-ball at P and a 1-ball in the compensating position. By this last move the minimum is reduced. This is impossible, therefore P cannot be a 2-space. The only case in which this argument breaks down is that in which S' coincides with S. When a ball has been moved from S to T there may not in this case be another one to move to T'. But, instead of moving a second ball from S to T' it is permissible to move the ball at T to P, and to obtain as before the impossible reduction of the minimum.

We now proceed as before; collecting between M_2 and M_4 those rows which have a 2-space within $L_1L_3M_3M_2$. In a similar way we define L_4. As before it follows that $L_1L_3M_3M_4$ and $L_3L_4M_1M_3$ are entirely 1-space and therefore empty. The rectangle $L_4L_2M_4M_2$ may be proved empty, and then L_5 and M_5 are found as before by collecting the filled columns of $L_3L_4M_4M_2$ to the left, and the filled rows of $L_4L_2M_3M_4$ to the top. This process is continued until it comes to a stop. At each stage it is possible to prove that the newly found diagonal rectangle is entirely 1-space if it occurs towards the top left-hand corner of the diagram, or empty if it occurs towards the bottom right-hand corner. As an example we now prove that $L_6L_4M_6M_4$ is empty.

If there is a 2-ball at some point E, then it must be possible to move it up to a 2-space at F, and to find another 2-ball at G which may be moved up to H, and a third at J which may be moved up to K. This multiple move may be compensated by moving a 1-ball from some point U to V. Now level with V there is the 2-space F', below which is a 2-ball at G'. We move this to H', then the ball from J' to K', and compensate the combined move by moving a 1-ball from U' to V'. As there are now 1-balls at V and at V' we can replace them by a 2-ball at F'' and a 1-ball at the appropriate point in $L_0L_1M_0M_1$. In this last move we reduce the minimum, and this is impossible. As before, the argument breaks down if K' lies below U, but in this case the impossible move is to move V directly to F'. The rectangle $L_6L_4M_6M_4$ is therefore empty.

The process which we have described cannot be continued indefinitely since some rows or columns are used up at each step and the number available is finite. The end is reached when one of the processes of separation (empty strips from filled, or strips containing 2-space from those which are entirely 1-space) breaks down because all the remaining strips are of one kind. In Fig. 4 the end is reached when no columns containing 2-balls are found in $L_5L_6M_6M_4$ or $L_6L_4M_5M_6$, which are therefore both empty. Since no complete row or column of the figure is empty the rectangle $L_5L_6M_5M_6$ must contain a 2-ball in every row or column, as shown.

The final result in Fig. 4 is that all the 1-balls are contained in $L_0L_5M_0M_6$, and also in $L_0L_6M_0M_5$, and that both these rectangles are entirely 1-space, and are rectangles of minimum content since the diagonally opposite rectangles of L_5LM_6M and L_6LM_5M are empty. We have therefore obtained a solution of the problem in integers. It is clearly the same solution as when the cell contents are not restricted to integers, since the minimum content of the rectangle of greatest minimum content is fixed by the boundary conditions alone, and does not depend on whether we specify integral or non-integral solutions.

The process by which Fig. 4 is built up can end in a number of different ways. An examination of the various possibilities shows that when it becomes impossible to subdivide the remaining sets of rows or columns the 1-balls are always found to be contained in a rectangle of greatest minimum content consisting entirely of 1-space. The theorem is therefore true in all cases.

(iv) *Infinite problem, general R* 1

The attempt to prove the theorem when the rows and columns are not given, and the rectangle may be divided up arbitrarily, leads to difficulties connected with the nature of the division of the rectangle into Regions 1 and 2. These difficulties may be avoided by restricting ourselves to divisions which satisfy the condition explained below.

If we divide the figure into an arbitrary number of rows and columns each elementary cell so produced must be assignable to one of three groups: Group A, those cells consisting entirely of 1-space except perhaps for points on their boundaries; Group B, those consisting partly of 1-space and partly of 2-space; Group C, those consisting entirely of 2-space, except perhaps for points on their boundaries. The area formed by the cells of Group A we shall call Region 1′, and the area formed by Groups A and B together we shall call Region 1″. These regions are groups of cells as in the finite problem.

We now assume that the minimum contents of the rectangles of greatest minimum content for Regions 1′ and 1″ differ by a quantity ϵ which can be made as small as we please by increasing the number of rows and columns. It seems likely that in most practical cases, where the subdivision is by means of one or more simple curves, and $A(x)$ and $B(y)$ possess finite derivatives at every point, it would be possible to show that this condition was fulfilled. If the derivatives exist, no line parallel to a side of the rectangle can carry a finite quantity of mass, and any finite number of such lines may be added to or deleted from an area without effect.

As a first step in proving the infinite case we divide up our rectangle into a finite number of rows and columns as just described and consider the minimum contents of the Regions 1′ and 1″. Since the rows and columns are now given and the regions are of the correct type we can construct equivalent finite problems by replacing the infinite boundary conditions by the finite number of conditions which specify the contents of these rows and columns. We know that we can find solutions of the equivalent problems. We have still to prove, however, that the contents of the rows and columns can be so arranged within the rows and columns themselves that the same solution is also a solution in the infinite case, i.e. is valid for a subdivision into any other set of rows and columns. In a subdivision like that which gives Region 1′, where each cell belongs completely to one region or the other, there is no difficulty in rearranging the content of an elementary rectangle in any way we please. It is easily seen that in such a case the contents of the rows and columns may be rearranged so as to satisfy the infinite boundary conditions also. Therefore the minimum content found for Region 1′ or 1″ by treating it as a problem in finite form is also the minimum content for Region 1′ or 1″ for the problem in infinite form.

Let the minimum content of Region 1 subject to the infinite boundary conditions be m. There must exist some rectangle R, contained in Region 1, whose minimum content m_R exceeds or equals that of any other rectangle contained in Region 1. Now the rectangles of greatest minimum content for Regions 1′ and 1″ differ in content by less than some quantity ϵ. Let these rectangles be denoted by R' and R'' and their minimum contents by m' and m''.

Now, apart from points confined to boundaries of rows or columns, Region 1′ is contained in Region 1, which is contained in Region 1″; therefore their minimum contents under the infinite conditions are in ascending order,

$$m' \leqslant m \leqslant m''. \tag{7}$$

But since Region 1′ is contained in Region 1 the rectangle of greatest minimum content in Region 1′ cannot have a greater content than the corresponding rectangle of Region 1, and this in turn cannot have a greater content than Region 1 itself, therefore

$$m' \leqslant m_R \leqslant m. \tag{8}$$

Remembering that m'' exceeds m' by less than ϵ we may write (7) and (8) as

$$m' \leqslant m_R \leqslant m \leqslant m' + \epsilon. \tag{9}$$

Since m_R and m are quantities independent of the subdivisions which give m' and ϵ, and since ϵ can be made as small as we please, we must have

$$m_R = m \tag{10}$$

which proves that, in this case also, the minimum content of Region 1 equals that of the rectangle of greatest minimum content within it.

(v) *Minimum within a minimum, finite non-integral problem*

Suppose that after the rectangle has been divided into Regions 1 and 2 we subdivide Region 1 into Regions 1_1 and 1_2, and require the minimum for Region 1_1 subject to the boundary conditions and to the extra condition that the content of Region 1 is maintained at its minimum. For brevity we shall refer to this as the conditional minimum for Region 1_1. We assume that the subdivision is such that any cell of Region 1 belongs completely either to Region 1_1 or to Region 1_2.

In Fig. 5 the rows and columns have first been arranged as in Fig. 2 in finding the minimum of Region 1; the dividing lines for this arrangement being lettered as before L_1, L_2, M_1 and M_2. Regions shaded at 45° from top right to bottom left are empty because of the minimum content of Region 1. The step curves XY and WZ, which form part of the boundary of this shaded region, will be explained below. There may, of course, be other empty regions which cannot be shaded because their positions are unknown; elements of Region 1 which lie outside the rectangle $L_0L_1M_0M_1$ are empty and there may be some of these to the right of XY and WZ.

It is convenient in stating the proof to re-define Region 1_1, to confine it to those parts of Region 1_1 which lie within the rectangle $L_0L_1M_0M_1$. This makes no difference to the answer we shall get for the minimum content, since any part of Region 1_1 which lies outside the rectangle $L_0L_1M_0M_1$ must be empty, and the minimum content of Region 1_1 must therefore lie entirely within rectangle $L_0L_1M_0M_1$. We therefore assume that all the cells of Region 1_1 lie within $L_0L_1M_0M_1$ and that all the rest of Region 1 is assigned to Region 1_2.

As before we assume that we have a solution of the problem and that by means of superposition any element which can contain a ball is made to do so. We now group the rows and columns of $L_0L_1M_0M_1$ as we did for the first minimum. Rows and columns containing a 1_1-ball are collected into M_0M_3 and L_0L_3. Rows M_4M_1 are those which have a 1_2-space within $L_0L_3M_3M_1$; columns L_4L_1 similarly have a 1_2-space in $L_3L_1M_0M_3$. We require also

a grouping of the rows and columns which pass through the rectangle $L_3L_4M_3M_4$. Those which are empty within this rectangle are grouped below M_5 and to the right of L_5. It cannot be assumed that every row and column in $L_3L_4M_3M_4$ is filled, because, although we assume

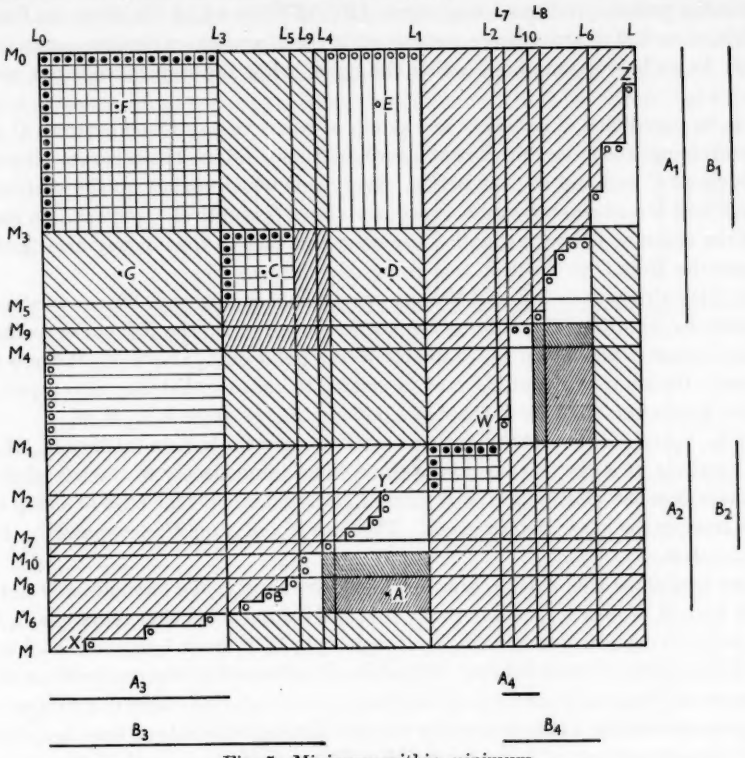

Fig. 5. Minimum within minimum.

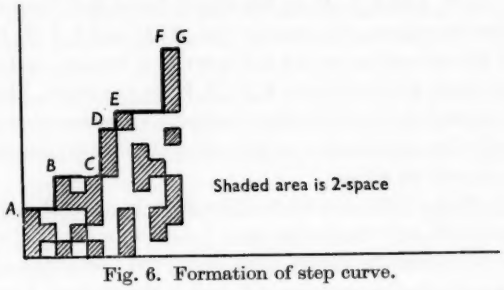

Fig. 6. Formation of step curve.

each complete row or column to be filled, there are other spaces available besides those which lie within the rectangle $L_3L_4M_3M_4$.

We now rearrange the rows M_2M in the following way, illustrated separately in Fig. 6. Consider the first column on the left. If any elements of it are 2-space we move to the bottom all the rows which contain these 2-spaces. The column will then be divided at A into an upper

part consisting entirely of 1_2-space and a lower part consisting entirely of 2-space. We then repeat the process with the second column and the rows above A, that is, we ignore the rows already brought down. The process is continued in this way until we reach L_1 in Fig. 5. It is clear that the process produces a step curve $ABCDEFG$ in which the elements forming the rising face of each step are entirely 2-space while the space above the step curve is entirely 1_2-space. As we have pointed out earlier the space below the step curve is not necessarily 2-space.

It is to be noted that if we select any group of columns, say those between C and G in Fig. 6, and change the order of the columns we shall get a different step curve. Nevertheless, it will begin at C and end at G as before. Every row which makes up the vertical height between C and G contains a 2-space, every row above G is entirely 1_2-space. No rearrangement of the columns between C and G can alter this fact so that in any arrangement the steps must rise from C to G within this group of columns.

In Fig. 5 the step curves XY and WZ are constructed in this way. Since every row below M_2 contains by definition at least one 2-space to the left of L_1, the XY curve reaches up to M_2 at some point to the left of L_1. Similarly W is a point on L_2 above M_1. We now see why the spaces to the left of XY and WZ may be shaded to indicate that they are empty because of the first minimum: they are 1-space and outside the rectangle $L_0L_1M_0M_1$.

Where L_3, L_4 and L_5 intersect the step curve XY we draw the horizontal lines M_6, M_7 and M_8. Strictly speaking, it is the intersection of the column immediately to the left of the L line which determines the intersection; e.g. where L_5 falls exactly on the edge of a step we must take the curve on the left to determine M_8. The lines M_3, M_4 and M_5 determine L_6, L_7 and L_8 in a similar way.

We may now show that certain areas in Fig. 5 are empty. We have already noted that rectangle $L_0L_3M_0M_3$ has a 1_1-ball in every row and column, and that rectangle $L_3L_5M_3M_5$ has a 1_2-ball in every row and column. Rectangle $L_0L_3M_4M_1$ has a 1_2-space in each row, and $L_4L_1M_0M_3$ a 1_2-space in each column. These are all indicated by the conventions of Fig. 2. The presence of 2-spaces in the risers of the step curves is also indicated in a similar fashion.

By arguments similar to those used for the first minimum it follows that $L_0L_4M_0M_3$ and $L_0L_3M_0M_4$ consist entirely of 1_1-space, and therefore apart from $L_0L_3M_0M_3$ they are empty. The rectangles $L_3L_1M_4M_1$ and $L_4L_1M_3M_1$ are empty because if filled they would enable us to perform impossible operations. Rectangles $L_3L_1M_6M$ and $L_6LM_3M_1$ are empty, because a ball in the first of these may be moved sideways to a 2-space under the step curve and compensated by removing a 1_1-ball from $L_0L_3M_0M_3$ to the right. This is the same sort of move as that which showed that $L_3L_1M_4M_1$ was empty. These spaces which have been shown to be empty by exactly the arguments used in finding the first minimum are shaded in Fig. 5 at 45° from top left to bottom right.

Now consider $L_4L_1M_8M_6$. If this is not empty let there be a 2-ball at A. This may be moved sideways to a 2-space at B, and there must be a 1_2-ball above it at C which may be moved to D. The ball at D, whatever its nature, may be moved upwards to a 1_2-space at E, and a 1_1-ball at F moved down to G, remaining a 1_1-ball in the process. The net result of this set of moves is that both minima are unaffected, but we have a 1_1-ball at G in a row which by definition it cannot occupy. Therefore, the move is impossible, and $L_4L_1M_8M_6$ must be empty. Similarly $L_8L_6M_4M_1$ is empty.

It would be convenient to be able to prove that $L_5L_4M_8M_6$ or $L_4L_1M_7M_8$ was empty in order to get a completely empty rectangle with an upper left-hand corner on the step curve.

We cannot do this because there are no 1_2-balls above the step curve between L_5 and L_4. We shall show in the next few paragraphs, however, that, if neither of these rectangles is empty, it is possible to rearrange the columns L_5L_4 and rows M_7M_8 and to find a line L_9 lying between L_5 and L_4 whose intersection with the rearranged step curve defines a line M_{10} for which the rectangle $L_9LM_{10}M$ is empty. A similar process applied to the rows M_5M_4 and columns L_7L_8 gives us the lines M_9 and L_{10} and an empty rectangle $L_{10}LM_9M$. The last elements of these empty regions are shaded steeply from top left to bottom right in Fig. 5.

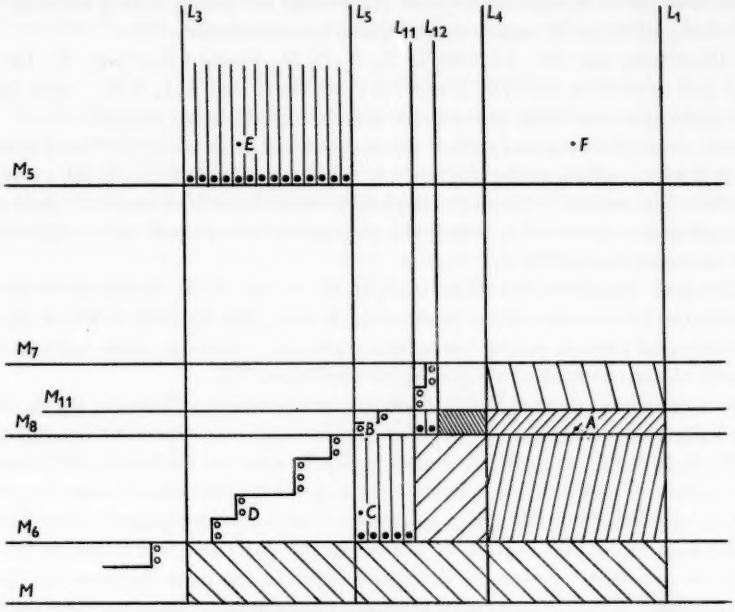

Fig. 7. Detail from Fig. 5.

Fig. 7 shows an enlarged version of the region between L_3 and L_1. In $L_3L_5M_8M_6$ every row contains a 2-space which is indicated by the fact that the step curve rises to M_8 within the rectangle. We next inquire whether the lower right-hand rectangle defined by L_5M_8 is empty or not. In Fig. 7 we have assumed that it is not empty, therefore some of the columns in $L_5L_4M_8M_6$ must be filled. We move these filled columns to the left thus defining their boundary line L_{11}. The remaining rectangle $L_{11}L_4M_8M_6$ is therefore empty. We next inquire whether there are any rows containing 2-spaces in $L_5L_{11}M_7M_8$. If there are none we shall have achieved our goal, since the step curve will then proceed horizontally along M_8 to its intersection with L_{11}, and we shall have an empty bottom right-hand rectangle $L_{11}LM_8M$ with its corner on the step curve. If, however, there are 2-spaces in $L_5L_{11}M_7M_8$ we collect them at the bottom, thus defining their upper boundary M_{11}. Between L_5 and L_{11} the step curve must therefore rise to M_{11}.

We may now prove that $L_4L_1M_{11}M_8$ is empty. If there were a 2-ball at A it might be moved sideways to a 2-space at B, below which there is a 2-ball at C which may be moved to a 2-space at D. The compensating move, a 1_2-ball from E to F, is exactly like the move C to D in Fig. 5, and may similarly be shown to lead to an impossibility if continued upwards.

We then inquire whether the rectangle $L_{11}L_4M_{11}M_8$ is empty. If it is we have reached our goal, but if it is not we collect the filled columns to the left as before and define the line L_{12}. From this point on we repeat the process. At each stage we inquire whether the rows above contain only 1-space or whether the columns on the right are empty. If the answer is affirmative we have reached our goal, if negative, we can reduce the remaining number of rows or columns. The process must come to an end because the number of rows and columns is finite. It can end in one of three ways, (1) in giving the sought for result, (2) because all the remaining rows up to M_7 contain 2-space, (3) because all the remaining columns up to L_4 contain 2-balls. If (2) or (3) occurs we can prove a contradiction.

Fig. 7 illustrates case (2). All rows in $L_{11}L_{12}M_7M_{11}$ contain 2-spaces. By the type of argument just applied to the $ABCD$ move we can show that $L_4L_1M_7M_{11}$ must be empty. But this contradicts our initial assumption that $L_4L_1M_7M_8$ is not empty.

If case (3) occurs it may be shown that the next step results in case (2). It may be seen from Fig. 7 that if when collecting the filled columns in the rectangle $L_{11}L_4M_{11}M_8$ every column has been found to contain a 2-ball the next step would have been case (2), since the rows containing 2-spaces (or in other words the step curve) must reach up to M_7 before L_4 is reached; see step curve in Fig. 5.

Thus, we may conclude that either $L_5L_4M_8M_6$ or $L_4L_1M_7M_8$ is empty or the process described above must at some stage produce the lines L_9 and M_{10} (Fig. 5) which intersect on the step curve and have an empty lower right quadrant. Similarly, there exist the lines M_9 and L_{10} with the corresponding properties for step curve WZ.

Consider now certain areas of Fig. 5. These are indicated by the lines A_1, A_2, A_3, A_4 drawn below and alongside the rectangle. The rectangles which are opposite these lines, that is, $L_0L_3M_0M_9$, $L_0L_3M_1M_6$, $L_2L_{10}M_0M_9$ and $L_2L_{10}M_1M_6$, contain all the 1_1-balls and consist either of 1_1-space or of space which is empty because of the first minimum (space shaded at 45° from top right to bottom left in Fig. 5). The space opposite the gaps in the lines, $L_3L_2M_9M_1$, $L_3L_2M_6M$, $L_{10}LM_9M_1$ and $L_{10}LM_6M$ are all empty. If the space opposite the lines is rearranged as a compact rectangle, the space opposite the gaps becomes the diagonally opposite rectangle. Thus the content of the space opposite the A lines is at its minimum. We have therefore shown that the conditional minimum content of Region 1_1 (i.e. subject to existence of minimum for Region 1) equals the unconditional minimum content of a rectangle which lies within a larger region obtained by adding to Region 1_1 the space which is necessarily empty because of the minimum of Region 1. The same thing is true of the rectangle defined by the B lines in Fig. 5.

Let the conditional minimum of Region 1_1 be denoted by m. Let the rectangle just discovered be denoted by R; its unconditional minimum is also m. Let the enlarged region be denoted by Region (1_1+0) and let its unconditional minimum content be m'.

Since Region (1_1+0) is empty apart from Region 1_1, its content is also the content of Region 1_1 which is m. Since no content can be less than the minimum content m' we have

$$m \geqslant m'. \tag{11}$$

But m is also the minimum content of rectangle R which lies entirely within the Region (1_1+0) and cannot therefore exceed the minimum for that region, therefore

$$m \leqslant m'. \tag{12}$$

Thus finally

$$m = m'. \tag{13}$$

It is also evident that R must be a rectangle of greatest minimum content for Region (1_1+0).

Thus, we have shown that the conditional minimum of Region 1_1 is the unconditional minimum content of the enlarged Region (1_1+0).

(vi) *Minimum within minimum: finite integral problem*

A proof can be constructed by combining the procedures of Figs. 4 and 5.

(vii) *Minimum within minimum: infinite problem*

No proof on the lines of that given for a single minimum has been found. It seems likely, however, that the theorem is true for most simple subdivisions of the rectangle.

(viii) *Extension to a third minimum*

When the division into regions is by means of simple and similar monotonic curves as in Fig. 3, some of the difficulties of Fig. 7 do not arise. In Fig. 3 Region 1 is all the space above DEF; Region 1_1 is the space above GHJ, and the third minimum region is the space above NPQ. In this simple case it may be shown that the minimum for the space above NPQ, subject to the two preceding minima, is given by a rule similar to that found for two minima. In this case empty space to be added to form the enlarged region is that which is necessarily empty because of the first two minima. This suggests that the theorem may be generalized to still further minima. It may well be valid for subdivisions other than those by similar monotonic curves.

(ix) *Further generalizations*

It might be expected that similar theorems would hold in more than two dimensions, the rectangle of greatest minimum content being replaced by an n-dimensional rectangular parallelepiped. But this is not so. The minimum content of a region cannot be less than that of a rectangular parallelepiped contained in it, but in some three-dimensional cases it has been shown to be greater.

4. EXAMPLES

(i) *Uniform distributions on a square*

An extremely simple case illustrates a peculiarity of the infinite problem: the appearance of line densities on the boundary of Region 1. Consider a unit square whose top left-hand diagonal half is Region 1. The distributions are uniform, i.e. $A(x) = x$, $B(y) = y$, and the diagonal is $x+y = 1$. The minimum content of a rectangle with its corner on the diagonal is zero, so that the minimum content is zero. It is easily seen that, if the square is divided into rows and columns, the entire mass must be distributed uniformly in the elementary half-squares just below the diagonal. In the infinite case this becomes a line density along the diagonal, but from our point of view it must be considered as lying in Region 2. By considering a parallel line just outside the diagonal boundary we may see that the content of Region 1 may be made as small as we please without the formation of a line density on the diagonal.

(ii) *Normal probability distribution, minimum within a circle*

In a normal standardized bivariate probability distribution in x and y, centred on $x = y = 0$, the probability in any strip of width dx is given by

$$dp = \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}x^2} dx \tag{14}$$

and similarly for dy. Let us suppose that the boundary conditions for our problem are in fact given by (14), that is, the distribution is at least compatible with the normal distribution. If the distribution were in fact normal we know that the probability p in a circle of radius r with centre at the origin would be given by

$$p = 1 - e^{-\frac{1}{2}r^2}. \tag{15}$$

We then inquire what is the minimum probability within the circle of radius r which is compatible with (14).

The rectangle of greatest minimum content is a square inscribed in the circle. If the halfside of the square is denoted by d, the minimum content m of the square is given by

$$m = \sqrt{\frac{2}{\pi}} \int_{-d}^{d} e^{-\frac{1}{2}x^2} dx - 1. \tag{16}$$

Both m and p are given in Table 1 for a range of radii. It may be noted that the change over from zero content to positive content takes place where

$$r = 0{\cdot}954, \tag{17}$$

i.e. a circle whose radius is almost equal to the standard deviation can be empty without making it impossible to satisfy the boundary conditions appropriate to a normal distribution.

Minimum within minimum

Only a single case has been worked out. The first minimum is for a circle of radius 2 and the second for the concentric circle of radius 1. It was found that the conditional minimum of the inner circle was 0·233, whereas the unconditional minimum was only 0·041.

Table 1. *Minimum content m of a circle of radius r compared with its content p in a normal distribution*

r	m	p	r	m	p	r	m	p
0·2	0·0	0·020	1·2	0·208	0·513	2·2	0·760	0·911
0·4	0·0	0·077	1·4	0·356	0·625	2·4	0·821	0·944
0·6	0·0	0·165	1·6	0·484	0·722	2·6	0·868	0·966
0·8	0·0	0·274	1·8	0·594	0·802	2·8	0·905	0·980
1·0	0·041	0·393	2·0	0·685	0·865	3·0	0·932	0·989

The work described here was carried out at the Road Research Laboratory of the Department of Scientific and Industrial Research; and is published by permission of the Director of Road Research.

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FURTHER CONTRIBUTIONS TO MULTIVARIATE CONFIDENCE BOUNDS*

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SUMMARY. In this paper the implications of certain results obtained in earlier papers (Roy & Bose, 1953*b*; Roy, 1954*a*, *b*, 1956) on confidence bounds on parametric functions connected with multivariate normal populations are fully worked out. This leads to a number of confidence bounds, expected to be useful, but hitherto unnoticed, on the characteristic roots connected with (i) one population dispersion matrix, (ii) two population dispersion matrices, (iii) the regression matrix of a p set on a q set, and (iv) the multivariate linear hypothesis on means, including, in particular, the problem of discriminant analysis. Some examples are given in the last section of the paper to illustrate the use of the techniques presented in the earlier sections.

1. CONFIDENCE BOUNDS ON ROOTS CONNECTED WITH ONE DISPERSION MATRIX

In conformity with the notation of previous papers, we consider a p-dimensional normal variate with dispersion matrix Σ and mean vector $\boldsymbol{\xi}$, $N(\boldsymbol{\xi}, \Sigma)$. The characteristic roots of Σ are represented by Θ. The dispersion matrix in a sample of size $(n+1)$ is written S and its characteristic roots represented by θ. The smallest and largest roots we denote respectively by θ_1 and θ_p, adjoining a suffix α to values between which they lie with probability $1-\alpha$, writing for example

$$P(\theta_{1\alpha} \leqslant \theta_1 \leqslant \theta_p \leqslant \theta_{2\alpha} \mid \Sigma) = 1-\alpha.$$

With a slight change in notation, let us denote the characteristic roots of a matrix M by $c(M)$ and the smallest and largest roots respectively by $c_{\min}(M)$ and $c_{\max}(M)$.

The statement (3·1·2) given in Roy (1954*a*) is exactly equivalent to

$$n\theta^{-1}_{1\alpha}(p,n)\,\mathbf{a}'S\mathbf{a} \geqslant \mathbf{a}'\Sigma\mathbf{a} \geqslant n\theta^{-1}_{2\alpha}(p,n)\,\mathbf{a}'S\mathbf{a} \ldots \tag{1·1}$$

for all non-null $\mathbf{a}(p \times 1)$'s, that is, to

$$\lambda_1 \frac{\mathbf{a}'S\mathbf{a}}{\mathbf{a}'\mathbf{a}} \geqslant \frac{\mathbf{a}'\Sigma\mathbf{a}}{\mathbf{a}'\mathbf{a}} \geqslant \lambda_2 \frac{\mathbf{a}'S\mathbf{a}}{\mathbf{a}'\mathbf{a}}, \tag{1·2}$$

where λ_1 and λ_2 stand respectively for $n\theta^{-1}_{1\alpha}(p,n)$ and $n\theta^{-1}_{2\alpha}(p,n)$.

Choosing $\mathbf{a}$ so as to minimize $\mathbf{a}'\Sigma\mathbf{a}/(\mathbf{a}'\mathbf{a})$, we observe that the second part of the inequality (1·2) implies that $\lambda_2\theta_1 \leqslant \Theta_1$; and choosing $\mathbf{a}$ so as to minimize $\mathbf{a}'S\mathbf{a}/(\mathbf{a}'\mathbf{a})$, we notice that the first part of the inequality implies that $\Theta_1 \leqslant \lambda_1\theta_1$. Likewise, choosing $\mathbf{a}$ so as to maximize $\mathbf{a}'S\mathbf{a}/(\mathbf{a}'\mathbf{a})$, we note that the second part of the inequality implies that $\lambda_2\theta_p \leqslant \Theta_p$; and choosing $\mathbf{a}$ so as to maximize $\mathbf{a}'\Sigma\mathbf{a}/(\mathbf{a}'\mathbf{a})$, we have that (1·2) implies that $\Theta_p \leqslant \lambda_1\theta_p$. Thus (1·2) implies the inequalities

$$\left.\begin{aligned} \lambda_1 c_{\min}(S) \geqslant c_{\min}(\Sigma) \geqslant \lambda_2 c_{\min}(S) \\ \lambda_1 c_{\max}(S) \geqslant c_{\max}(\Sigma) \geqslant \lambda_2 c_{\max}(S) \end{aligned}\right\}. \tag{1·3}$$

We note that (1·3) has a confidence coefficient $\geqslant 1-\alpha$.

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† Now with the Procter and Gamble Company.

Going back to (1·2) let us take $\mathbf{a}(p \times 1)$ such that the ith component is zero. Then arguing in a similar manner, we observe that (1·2) implies

$$\left.\begin{aligned} \lambda_1 c_{\min}(S^{(i)}) \geqslant c_{\min}(\Sigma^{(i)}) \geqslant \lambda_2 c_{\min}(S^{(i)}) \\ \lambda_1 c_{\max}(S^{(i)}) \geqslant c_{\max}(\Sigma^{(i)}) \geqslant \lambda_2 c_{\max}(S^{(i)}) \end{aligned}\right\} \tag{1·4}$$

for $i = 1, 2, \ldots, p$, where $S^{(i)}, \Sigma^{(i)}$ stand respectively for the 'truncated' sample and population dispersion matrices obtained by cutting out the ith variate. Likewise, if we take an $\mathbf{a}(p \times 1)$ such that the ith and jth $(i \neq j)$ components are zero and then argue in a similar manner, we observe that (1·2) also implies

$$\left.\begin{aligned} \lambda_1 c_{\min}(S^{(i,j)}) \geqslant c_{\min}(\Sigma^{(i,j)}) \geqslant \lambda_2 c_{\min}(S^{(i,j)}), \\ \lambda_1 c_{\max}(S^{(i,j)}) \geqslant c_{\max}(\Sigma^{(i,j)}) \geqslant \lambda_2 c_{\max}(S^{(i,j)}) \end{aligned}\right\} \tag{1·5}$$

for $i \neq j = 1, 2, \ldots, p$, where $S^{(i,j)}$, $\Sigma^{(i,j)}$, stand respectively, for the truncated sample and population dispersion matrices obtained by cutting out the ith and jth variates. We can continue this process on to the stage of cutting out any $(p-1)$ variates, that is, retaining any one variate. It is seen that (1·2) implies a pair of statements (1·3), and also p pairs of statements like (1·4), $\binom{p}{2}$ pairs of statements like (1·5), and so on down to $\binom{p}{p-1}$, i.e. p statements involving only one variate. All such statements will thus have a joint confidence coefficient $\geqslant 1-\alpha$, and will provide us, from a certain standpoint, with a complete analysis of what the psychologists call the problem of principal components.

2. Confidence bounds on roots connected with two population dispersion matrices

Similarly, if we have two p dimensional normal variates with mean vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ and dispersion matrices Σ_1 and Σ_2, we may write equation (3·2·1) of Roy (1954*a*), putting $\lambda_1 = (n_1/n_2)\,\theta_{1\alpha}^{-1}(p, n_1, n_2)$ and $\lambda_2 = (n_1/n_2)\,\theta_{2\alpha}^{-1}(p, n_1, n_2)$, as

$$\lambda_1 \geqslant \text{ all } c(S_2(\mu')^{-1} D_{\sqrt{\gamma_i}} \mu' S_1^{-1} \mu D_{\sqrt{\gamma_i}} \mu^{-1}) \geqslant \lambda_2, \tag{2·1}$$

where γ_i's are $c(\Sigma_1 \Sigma_2^{-1})$'s (with $i = 1, 2, \ldots, p$). We next recall that

$$\text{all non-zero } c(A(p \times q)\, B(q \times p)) = \text{the non-zero } c(B(q \times p)\, A(p \times q))$$

and that $c(S_1 S_2^{-1})$ are invariant under a transformation: $S_1 = A S_1^* A'$ and $S_2 = A S_2^* A'$, where A is any non-singular matrix. Putting $A = \mu^{-1}$, we rewrite (2·1), without any loss of generality, for our purpose, in the canonical form

$$\lambda_1 \geqslant \text{ all } c(S_2 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geqslant \lambda_2, \tag{2·2}$$

or

$$\lambda_1 \geqslant \text{ all } c(S_2 S_1^{-1} S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geqslant \lambda_2$$

or

$$\lambda_1 \frac{\mathbf{a}' S_1 S_2^{-1} \mathbf{a}}{\mathbf{a}'\mathbf{a}} \geqslant \frac{\mathbf{a}'(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}})\,\mathbf{a}}{\mathbf{a}'\mathbf{a}} \geqslant \lambda_2 \frac{\mathbf{a}' S_1 S_2^{-1} \mathbf{a}}{\mathbf{a}'\mathbf{a}}, \tag{2·3}$$

for all non-null $\mathbf{a}(p \times 1)$'s. Now choosing $\mathbf{a}$ so as to maximize the middle term of (2·3), we note that the left part of the inequality (2·3) implies that $\lambda_1 c_{\max}(S_1 S_2^{-1}) \geqslant c_{\max}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}})$; and choosing $\mathbf{a}$ so as to minimize the middle term of (2·3), we note that the right part of (2·3) implies that $c_{\min}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geqslant \lambda_2 c_{\min}(S_1 S_2^{-1})$. Thus (2·3) implies

$$\lambda_1 c_{\max}(S_1 S_2^{-1}) \geqslant c_{\max}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geqslant c_{\min}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geqslant \lambda_2 c_{\min}(S_1 S_2^{-1}). \tag{2·4}$$

It has been shown in Roy (1956) that

$$c_{\max}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}) \geqslant \text{ all } c(D_{\gamma_i}) \geqslant c_{\min}(S_1 D_{\sqrt{\gamma_i}} S_1^{-1} D_{\sqrt{\gamma_i}}). \tag{2·5}$$

Thus it is seen that (2·3) implies

$$\lambda_1 c_{\max}(S_1 S_2^{-1}) \geqslant \text{ all } c(\Sigma_1 \Sigma_2^{-1}) \geqslant \lambda_2 c_{\min}(S_1 S_2^{-1}), \tag{2·6}$$

which, therefore, is a confidence statement with a confidence coefficient $\geqslant 1-\alpha$, since (2·3) has the confidence coefficient $1-\alpha$. (2·6) is derived in a slightly different way in Roy (1956).

We now go back to (2·3) and, as in the previous section, take $\mathbf{a}(p \times 1)$ such that the ith component is zero, argue the same way as from (2·3) to (2·6) and end up by observing that (2·3) also implies

$$\lambda_1 c_{\max}(S_1^{(i)} S_2^{(i)-1}) \geqslant \text{ all } c(\Sigma_1^{(i)} \Sigma_2^{(i)-1}) \geqslant \lambda_2 c_{\min}(S_1^{(i)} S_2^{(i)-1}), \tag{2·7}$$

where $S_1^{(i)}$, $S_2^{(i)}$, $\Sigma_1^{(i)}$ and $\Sigma_2^{(i)}$ have the same meaning as in the previous section. Likewise, as in the previous section, we note that (2·3) also implies

$$\lambda_1 c_{\max}(S_1^{(i,j)} S_2^{(i,j)-1}) \geqslant \text{ all } c(\Sigma_1^{(i,j)} \Sigma_2^{(i,j)-1}) \geqslant \lambda_2 c_{\min}(S_1^{(i,j)} S_2^{(i,j)-1}), \tag{2·8}$$

and so on till we reach the stage where any $(p-1)$ variates have been cut out, i.e. any one variate has been retained, which gives us just the confidence bounds on variance ratios in the univariate case. We have thus, with a joint confidence coefficient $\geqslant 1-\alpha$, a confidence statement (2·6), p confidence statements like (2·7), $\binom{p}{2}$ confidence statements like (2·8), and so on. This again, from a certain standpoint, provides part of the analysis of a problem which occurs in the multivariate generalization of the customary variance components analysis in univariate analysis of variance and covariance.

3. Confidence bounds on roots connected with the regression matrix $\beta(p \times q)$ of a p set on a q set in a $(p+q)$-variate normal distribution

Let the population be denoted

$$N\begin{bmatrix} p\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}, & \Sigma = p\begin{bmatrix}\Sigma_{11} & \Sigma_{12}\\ \Sigma_{12}' & \Sigma_{22}\end{bmatrix} \\ 1 & p \quad q \end{bmatrix}$$

so that the p set has mean vector ξ_1 and the dispersion matrix Σ_{11} and the q set has mean vector ξ_2 and dispersion matrix Σ_{22}. Σ_{12} is the matrix of covariances between the p set and the q set and we define $\beta(p \times q) = \Sigma_{12}\Sigma_{22}^{-1}$. Let the sample dispersion matrix based on a sample of size $(n+1)$ be written $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}' & S_{22}\end{bmatrix}$ and let $B(p \times q) = S_{12} S_{22}^{-1}$ be the sample regression matrix of the p set on the q set. Also, let us write

$$S_{1\cdot 2}(p \times p) = S_{11} - S_{12} S_{22}^{-1} S_{12}'.$$

Setting, $\lambda^2 = \theta_\alpha/(1-\theta_\alpha)$ we can now rewrite (4·5) of Roy (1954a) as

$$\text{all } c[(B-\beta)(B'-\beta')] \leqslant \lambda^2 c_{\max}(S_{1\cdot 2})\, c_{\max}(S_{22}^{-1}) \tag{3·1}$$

with a confidence coefficient $\geqslant 1-\alpha$. We recall the Lemmas C_i' and E_i' of Roy (1954a), viz. that (i) the statement '$g_1 \leqslant$ all $c(M) \leqslant g_2$(for a $p \times p$ real matrix M with real roots)' is

equivalent to the statement '$g_1 \leqslant \mathbf{d}'(1\times p)\,M(p\times p)\,\mathbf{d}(p\times 1) \leqslant g_2$ (for all arbitrary unit vectors $\mathbf{d}$)'; and that (ii) the statement '$\mathbf{x}'(1\times q)\,\mathbf{x}(q\times 1) \leqslant h(>0)$' is equivalent to the statement '$(\mathbf{x}'(1\times q)\,\mathbf{d}(q\times 1)) \leqslant \sqrt{h}$ (for all arbitrary unit vectors $\mathbf{d}$)'. Using these two lemmas, we obtain from (3·1) the equivalent set of confidence statements,

$$\mathbf{d}_1' B\mathbf{d}_2 - \lambda c_{\max}^{\frac{1}{2}}(S_{1\cdot 2})\, c_{\max}^{\frac{1}{2}}(S_{22}^{-1}) \leqslant \mathbf{d}_1'\beta\mathbf{d}_2 \leqslant \mathbf{d}_1' B\mathbf{d}_2 + \lambda c_{\max}^{\frac{1}{2}}(S_{1\cdot 2})\, c_{\max}^{\frac{1}{2}}(S_{22}^{-1}), \tag{3·2}$$

for all unit vectors $\mathbf{d}_1(p\times 1)$ and $\mathbf{d}_2(q\times 1)$, with a confidence coefficient $\geqslant 1-\alpha$. Going back to the above lemmas again we notice that, with respect to variation over $\mathbf{d}_1$ and $\mathbf{d}_2$, the maximum values of $\mathbf{d}_1' B\mathbf{d}_2$ and $\mathbf{d}_1'\beta\mathbf{d}_2$ are respectively $c_{\max}^{\frac{1}{2}}(BB')$ and $c_{\max}^{\frac{1}{2}}(\beta\beta')$. Now, first choosing $\mathbf{d}_1$ and $\mathbf{d}_2$ so as to maximize $\mathbf{d}_1' B\mathbf{d}_2$ and then choosing $\mathbf{d}_1$ and $\mathbf{d}_2$ so as to maximize $\mathbf{d}_1'\beta\mathbf{d}_2$ and arguing in the same way as in the previous sections we note that (3·2) implies

$$c_{\max}^{\frac{1}{2}}(BB') - \lambda c_{\max}^{\frac{1}{2}}(S_{1\cdot 2})\, c_{\max}^{\frac{1}{2}}(S_{22}^{-1}) \leqslant c_{\max}^{\frac{1}{2}}(\beta\beta') \leqslant c_{\max}^{\frac{1}{2}}(BB'_{\max}) + \lambda c_{\max}^{\frac{1}{2}}(S_{1\cdot 2})\, c_{\max}^{\frac{1}{2}}(S_{22}^{-1}), \tag{3·3}$$

which, therefore, is a confidence statement with a confidence coefficient $\geqslant 1-\alpha$. We now rewrite (4·4) of Roy (1954*a*) in the equivalent form

$$\frac{\mathbf{a}'(B-\beta)\,S_{22}(B'-\beta')\,\mathbf{a}}{\mathbf{a}'S_{1\cdot 2}\mathbf{a}} \leqslant \lambda^2, \tag{3·4}$$

for all non-null $\mathbf{a}(p\times 1)$'s, which is a confidence statement with a confidence coefficient $1-\alpha$. This means that (3·4), with a probability $1-\alpha$, implies (3·3), with a probability $\geqslant 1-\alpha$. As in the previous sections, take $\mathbf{a}(p\times 1)$ such that the ith component is zero, define $S_{1\cdot 2}^{(i)}$, $B^{(i)}$ and $\beta^{(i)}$ as the 'truncated' matrices obtained by cutting out the ith variate of the p set, and observe that (3·4) also implies

$$\begin{aligned} c_{\max}^{\frac{1}{2}}(B^{(i)}B^{(i)'}) - \lambda c_{\max}^{\frac{1}{2}}(S_{1\cdot 2}^{(i)})\, c_{\max}^{\frac{1}{2}}(S_{22}^{-1}) &\leqslant c_{\max}^{\frac{1}{2}}(\beta^{(i)}\beta^{(i)'}) \\ &\leqslant c_{\max}^{\frac{1}{2}}(B^{(i)}B^{(i)'}) + \lambda c_{\max}^{\frac{1}{2}}(S_{1\cdot 2}^{(i)})\, c_{\max}^{\frac{1}{2}}(S_{22}^{-1}). \end{aligned} \tag{3·5}$$

Likewise, as in the previous sections, we observe that (3·4) also implies

$$\begin{aligned} c_{\max}^{\frac{1}{2}}(B^{(i,j)}B^{(i,j)'}) - \lambda c_{\max}^{\frac{1}{2}}(S_{1\cdot 2}^{(i,j)})\, c_{\max}^{\frac{1}{2}}(S_{22}^{-1}) &\leqslant c_{\max}^{\frac{1}{2}}(\beta^{(i,j)}\beta^{(i,j)'}) \\ &\leqslant c_{\max}^{\frac{1}{2}}(B^{(i,j)}B^{(i,j)'}) + \lambda c_{\max}^{\frac{1}{2}}(S_{1\cdot 2}^{(i,j)})\, c_{\max}^{\frac{1}{2}}(S_{22}^{-1}), \end{aligned} \tag{3·6}$$

and so on. We have thus, with a joint confidence coefficient $\geqslant 1-\alpha$, the statement (3·3), p statements like (3·5), $\binom{p}{2}$ statements like (3·6) and so on. This kind of result can be generalized by truncating the variates of the q set as well, but this will not be discussed here.

4. Confidence bounds on roots connected with multivariate linear hypothesis on means

4a. On $\boldsymbol{\xi}$ of $N(\boldsymbol{\xi}, \Sigma)$. We have a random sample of size $(n+1)$ and the sample mean vector is written as $\bar{\mathbf{x}}$ while the sample dispersion matrix is S. Setting $\lambda^2 = T_\alpha^2/n+1$, where T_α^2 is the upper α% point of Hotelling's T^2 distribution with p and $n+1-p$ degrees of freedom, we can rewrite (4·1·4) of Roy & Bose (1953*b*) as

$$\frac{\mathbf{a}'\bar{\mathbf{x}}}{(\mathbf{a}'\mathbf{a})^{\frac{1}{2}}} - \lambda\frac{(\mathbf{a}'S\mathbf{a})^{\frac{1}{2}}}{(\mathbf{a}'\mathbf{a})^{\frac{1}{2}}} \leqslant \frac{\mathbf{a}'\boldsymbol{\xi}}{(\mathbf{a}'\mathbf{a})^{\frac{1}{2}}} \leqslant \frac{\mathbf{a}'\bar{\mathbf{x}}}{(\mathbf{a}'\mathbf{a})^{\frac{1}{2}}} + \lambda\frac{(\mathbf{a}'S\mathbf{a})^{\frac{1}{2}}}{(\mathbf{a}'\mathbf{a})^{\frac{1}{2}}}, \tag{4·1}$$

for all non-null $\mathbf{a}(p\times 1)$'s. We recall that (4·1) is a confidence statement with a confidence coefficient $1-\alpha$. We recall also that the maximum values of $\mathbf{a}'\bar{\mathbf{x}}/(\mathbf{a}'\mathbf{a})^{\frac{1}{2}}$, $\mathbf{a}'\boldsymbol{\xi}/(\mathbf{a}'\mathbf{a})^{\frac{1}{2}}$,

$(\mathbf{a}'S\mathbf{a})^{\frac{1}{2}}/(\mathbf{a}'\mathbf{a})^{\frac{1}{2}}$, with respect to variation over $\mathbf{a}$'s, are respectively $(\bar{\mathbf{x}}'\bar{\mathbf{x}})^{\frac{1}{2}}$, $(\boldsymbol{\xi}'\boldsymbol{\xi})^{\frac{1}{2}}$ and $c^{\frac{1}{2}}_{\max}(S)$, then we reason in the same way as in the previous sections and deduce that (4·1) implies

$$[\bar{\mathbf{x}}'\bar{\mathbf{x}}]^{\frac{1}{2}} - \lambda c^{\frac{1}{2}}_{\max}(S) \leqslant [\boldsymbol{\xi}'\boldsymbol{\xi}]^{\frac{1}{2}} \leqslant [\bar{\mathbf{x}}'\bar{\mathbf{x}}]^{\frac{1}{2}} + \lambda c^{\frac{1}{2}}_{\max}(S), \tag{4·2}$$

which is thus a confidence statement with a confidence coefficient $\geqslant 1-\alpha$. Arguing as in previous sections and using the same notation as before for 'truncated' $\bar{\mathbf{x}}$, $\boldsymbol{\xi}$ and S obtained by cutting out the ith variate, the ith and jth variates ($i \neq j$), and so on, we have with a joint confidence coefficient $\geqslant 1-\alpha$, in addition to (4·2), p statements like

$$[\bar{\mathbf{x}}^{(i)\prime}\bar{\mathbf{x}}^{(i)}]^{\frac{1}{2}} - \lambda c^{\frac{1}{2}}_{\max}(S^{(i)}) \leqslant [\boldsymbol{\xi}^{(i)\prime}\boldsymbol{\xi}^{(i)}]^{\frac{1}{2}} \leqslant [\mathbf{x}^{(i)\prime}\bar{\mathbf{x}}^{(i)}]^{\frac{1}{2}} + \lambda c^{\frac{1}{2}}_{\max}(S^{(i)}), \tag{4·3}$$

$\binom{p}{2}$ statements like

$$[\bar{\mathbf{x}}^{(i,j)\prime}\bar{\mathbf{x}}^{(i,j)}]^{\frac{1}{2}} - \lambda c^{\frac{1}{2}}_{\max}(S^{(i,j)}) \leqslant [\boldsymbol{\xi}^{(i,j)\prime}\boldsymbol{\xi}^{(i,j)}]^{\frac{1}{2}} \leqslant [\bar{\mathbf{x}}^{(i,j)\prime}\mathbf{x}^{(i,j)}]^{\frac{1}{2}} + \lambda c^{\frac{1}{2}}_{\max}(S^{(i,j)}), \tag{4·4}$$

and so on down to the stage of cutting out any $(p-1)$ variates, i.e. retaining any one variate.

4*b*. *Some observations on multivariate linear hypothesis on means.* Confidence bounds connected with univariate and multivariate linear hypothesis on means are discussed respectively in Chapters 15 and 16 of Roy (1954*b*). In this section we shall first set up a physically more general hypothesis and then discuss the associated confidence bounds.

Let $X(n \times p)$ consist of n row vectors $\mathbf{x}_i'(1 \times p)$ (with $i = 1, 2, \ldots, n$) which are independently distributed, such that $\mathbf{x}_i$ is $N[E(\mathbf{x}_i), \Sigma]$ and let $E(X)\,(n \times p) = A(n \times m)\,\xi(m \times p)$, where $m < n$ and rank $(A) = r \leqslant m$. Let $A_1(n \times r)$ be a basis of A and let us write (as we can, without any loss of generality) $A(n \times m) = \underset{r}{[A_1}\quad \underset{m-r}{A_2}\,]\,n$ and let us rewrite the expectation condition as

$$\underset{p}{E(X)}\,n = n\underset{r}{[A_1}\quad \underset{m-r}{A_2}\,]\begin{bmatrix}\xi_1\\ \xi_2\end{bmatrix}\begin{matrix}r\\ m-r\end{matrix}. \tag{4·5}$$
$$\qquad\qquad\qquad

However, in most realistic problems, the C matrix of the hypothesis is given in a form such that the last rows are absent and we can, therefore, without any essential loss of generality, replace (4·6) by

$$\begin{matrix} & r & m-r \\ s[& C_1 & C_2 \end{matrix}\ ] \begin{bmatrix} \xi_1 (r \times p) \\ \xi_2 (m-r \times p) \end{bmatrix} M(p \times u) = 0, \tag{4·8}$$

and (4·7) by

$$C_2 = C_1 (A_1' A_1)^{-1} A_1' A_2. \tag{4·9}$$

We now go back to X and observe that $X(n \times p)\, M(p \times u)\, [= X^*(n \times u)$, say] consists of n rows of independently distributed vectors $\mathbf{x}_i'(1 \times p)\, M(p \times u)\, [= \mathbf{x}_i^{*\prime}(1 \times u)$, say] such that $\mathbf{x}_i^*$ is $N[E(\mathbf{x}_i^*),\ M'\Sigma M]$, i.e. $N[E(\mathbf{x}_i^*),\ \Sigma^*]$, say, and that

$$E(X^*) = [A_1 \quad A_2] \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} M = [A_1 \quad A_2] \begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix}, \tag{4·10}$$

say, where

$$\begin{matrix} & r \\ m-r & \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \\ & p \end{matrix} M(p \times u) = \begin{matrix} & r \\ m-r & \begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix} \\ & u \end{matrix}. \tag{4·11}$$

The H_0 of (4·8) can now be rewritten as

$$H_0\colon \quad \begin{matrix} & r & m-r \\ s[& C_1 & C_2 \end{matrix}\ ] \begin{matrix} \begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix} & \begin{matrix} r \\ m-r \end{matrix} \\ u & \end{matrix} = 0 \tag{4·12}$$

and the alternative H to H_0 can be expressed as

$$H\colon \quad \begin{matrix} & r & m-r \\ s[& C_1 & C_2 \end{matrix}\ ] \begin{matrix} \begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix} & \begin{matrix} r \\ m-r \end{matrix} \\ u & \end{matrix} = \eta^*(s \times u). \tag{4·13}$$

We next recall (16·6·3) of Roy (1954*b*), viz.,

$$\begin{aligned} \mathbf{a}' X' A_1 (A_1' A_1)^{-1} C_{11}' \bar{U} \mathbf{b} - (\mathbf{a}' S \mathbf{a})^{\frac{1}{2}} [s c_\alpha(p, s, n-r)]^{\frac{1}{2}} &\leqslant \mathbf{a}' \eta' \bar{U} \mathbf{b} \\ &\leqslant \mathbf{a}' X' A_1 (A_1' A_1)^{-1} C_{11}\, \bar{U} \mathbf{b} + (\mathbf{a}' S \mathbf{a})^{\frac{1}{2}} [s c_\alpha(p, s, n-r)]^{\frac{1}{2}}, \end{aligned} \tag{4·14}$$

for all non-null $\mathbf{a}(p \times 1)$'s and all unit vectors $\mathbf{b}(s \times 1)$, and substitute $M'X'$, i.e. $X^{*\prime}$ for X', C_1 for C_{11}, $\mathbf{a}^*(u \times 1)$ for $\mathbf{a}(p \times 1)$, u for p and η^* for η. We then see that the confidence statement (4·14) is replaced by

$$\begin{aligned} \mathbf{a}^{*\prime} X^{*\prime} A_1 (A_1' A_1)^{-1} C_1' \bar{U} \mathbf{b} - (\mathbf{a}^{*\prime} S^* \mathbf{a}^*)^{\frac{1}{2}} [s c_\alpha(u, s, n-r)]^{\frac{1}{2}} \\ \leqslant \mathbf{a}^{*\prime} \eta^{*\prime} \bar{U} \mathbf{b} \leqslant \mathbf{a}^{*\prime} X^{*\prime} A_1 (A_1' A_1)^{-1} C_1' \bar{U} \mathbf{b} + (\mathbf{a}^{*\prime} S^* \mathbf{a}^*)^{\frac{1}{2}} [s c_\alpha(u, s, n-r)]^{\frac{1}{2}}, \end{aligned} \tag{4·15}$$

for all non-null $\mathbf{a}^*(u \times 1)$ and all unit vectors $\mathbf{b}(s \times 1)$, where η^* is given by (4·13), $\begin{bmatrix} \xi_1^* \\ \xi_2^* \end{bmatrix}$ by (4·11), X^* stands for XM, and where

$$\bar{U} \bar{U}' = [C_1 (A_1' A_1)^{-1} C_1']^{-1} \tag{4·16}$$

and

$$(n-r)\, S^*(u \times u) = M'(u \times p)\, X'(p \times n) [I(n) - A_1 (A_1' A_1)^{-1} A_1'] X(n \times p)\, M(p \times u). \tag{4·17}$$

We note that (4·15) is a set of confidence statements, with a joint confidence coefficient $1-\alpha$, on bilinear compounds of η^*, where η^*, defined in (4·13), may be regarded as measuring the deviation from the null hypothesis H_0.

4c. *Further consequences of* (4·15). Starting from (4·15) and arguing in the same manner as in §3 and setting $c_\alpha(u, s, n-r) = c_\alpha$ (say), we note that (4·15) implies

$$c_{\max}^{\frac{1}{2}}[X^{*\prime}A_1(A_1'A_1)^{-1}C_1'\bar{U}\bar{U}'C_1(A_1'A_1)^{-1}A_1'X^*] - [sc_\alpha]^{\frac{1}{2}}c_{\max}^{\frac{1}{2}}(S^*) \leqslant c_{\max}^{\frac{1}{2}}[\eta^{*\prime}(\bar{U}\bar{U}')\eta^*]$$
$$\leqslant c_{\max}^{\frac{1}{2}}[X^{*\prime}A_1(A_1'A_1)^{-1}C_1'\bar{U}\bar{U}'C_1(A_1'A_1)^{-1}A_1'X^*] + [sc_\alpha]^{\frac{1}{2}}c_{\max}^{\frac{1}{2}}(S^*), \quad (4\cdot18)$$

or substituting for $\bar{U}\bar{U}'$ from (4·16),

$$c_{\max}^{\frac{1}{2}}(sS^{**}) - [s\alpha]^{\frac{1}{2}}c_{\max}^{\frac{1}{2}}(S^*) \leqslant c_{\max}^{\frac{1}{2}}[\eta^{*\prime}[C_1(A_1'A_1)^{-1}C_1']^{-1}\eta^*] \leqslant c_{\max}^{\frac{1}{2}}(sS^{**}) + [sc_\alpha]^{\frac{1}{2}}c_{\max}^{\frac{1}{2}}(S^*), \quad (4\cdot19)$$

where the matrix due to the hypothesis, i.e. sS^{**} is given by

$$sS^{**} = M'X'A_1(A_1'A_1)^{-1}C_1'[C_1(A_1'A_1)^{-1}C_1']^{-1}C_1(A_1'A_1)^{-1}A_1'XM, \quad (4\cdot20)$$

and the matrix due to the error, i.e. $(n-r)S^*$ is given by (4·17). Notice that (4·19) is a confidence statement with a confidence coefficient $\geqslant 1-\alpha$ and that the middle term of (4·19) is zero if, and only if, the hypothesis H_0 is true. For $p = 1$, $M(p \times u)$ will drop out (except for a trivial scalar factor, since $u \leqslant p$) and we shall have the univariate problem where $c_{\max}(sS^{**})$ will be replaced by just the sum of squares due to the hypothesis, $c_{\max}(S^*)$ by just the error mean sum of squares and

$$c_{\max}[\eta^{*\prime}[C_1(A_1'A_1)^{-1}C_1']^{-1}\eta^*] \quad \text{by just the scalar} \quad \boldsymbol{\eta}^{*\prime}(C_1(A_1'A_1)^{-1}C_1')^{-1}\boldsymbol{\eta}^*.$$

Starting from (4·15) and reasoning in exactly the same way as in §§3 and 4*a* we see that (4·15) also implies, in addition to (4·19), u statements like (4·19) involving 'truncated' $S^{*(i)}$, $S^{**(i)}$ and $\eta^{*(i)}$ obtained by cutting out any ith variate, $\binom{u}{2}$ statements like (4·19) involving 'truncated' $S^{*(i,j)}$, $S^{**(i,j)}$ and $\eta^{**(i,j)}$ obtained by cutting out any pair of ith and jth variates (with $i \neq j$), and so on. These latter confidence statements will thus have a joint confidence coefficient $\geqslant 1-\alpha$.

It may be noted that the problem discussed in §4*a* is really a special case of the one discussed in §4*c*; nevertheless, for expository purposes, it is worthwhile to discuss first a simple problem like the one in §4*a* and then take up the most general one in §4*c*.

5. EXAMPLES TO ILLUSTRATE THE TECHNIQUES OF §§3 AND 4

5·1. To illustrate §4 or, more properly speaking, the method developed in §4*c* for the general problem of multivariate analysis of variance of means, we use the data from a numerical example discussed in a standard textbook (Rao, 1952). The data consist of three kinds of physical measurements on 140 school children, in more or less the same age group, from six different high schools. To translate into the set-up of this paper, let $\boldsymbol{\xi}_i(3 \times 1)$, for $i = 1, 2, \ldots, 6$, denote the population mean vectors and let

$$\xi(6 \times 3) = \begin{bmatrix} \boldsymbol{\xi}_1' \\ \vdots \\ \boldsymbol{\xi}_6' \end{bmatrix} \begin{matrix} 1 \\ \\ 1 \end{matrix}.$$
$$3$$

Then, if $X(n \times 3)$, where n = total sample size = 140, be the observation matrix, we can write

$$E(X) = A(n \times 6)\,\xi(6 \times 3), \quad (5\cdot1\cdot1)$$

where
$$A(n\times 6)=\begin{matrix}\\ n_1\\ \\ \\ \\ n_6\\ \\ \end{matrix}\begin{bmatrix}1&0&0&0&.&.&.&0\\ .&.&.&.&.&.&.&.\\ 1&0&0&0&.&.&.&0\\ \hline .&.&.&.&.&.&.&.\\ .&.&.&.&.&.&.&.\\ \hline 0&0&0&0&.&.&.&1\\ .&.&.&.&.&.&.&.\\ 0&0&0&0&.&.&.&1\end{bmatrix},\quad n_1+\ldots+n_6=140$$

and $r(A)=6$.

Also H_0: $\xi_1=\ldots=\xi_6 \langle\Longrightarrow\rangle C\xi=0(5\times 3)$, where

$$C(5\times 6)=\begin{bmatrix}1&0&.&.&0&-1\\ 0&1&.&.&0&-1\\ .&.&.&.&.&.\\ 0&0&.&.&1&-1\end{bmatrix}$$

so that $r(C)=5$. Furthermore H: $C\xi=\eta(5\times 3)$. Hence we have $p=3, n=140, r=6, s=5$. Next we take over from Rao's book the 'between' product moment and 'within' product moment matrices given respectively by

$$sS^{**}(3\times 3)=\begin{bmatrix}752{\cdot}0&214{\cdot}2&521{\cdot}3\\ &151{\cdot}3&401{\cdot}2\\ &&1612{\cdot}7\end{bmatrix}$$

and
$$(n-r)\,S^{*}(3\times 3)=\begin{bmatrix}12809{\cdot}3&1003{\cdot}7&2671{\cdot}2\\ &1499{\cdot}6&4123{\cdot}6\\ &&21009{\cdot}6\end{bmatrix}.$$

According to the set-up of this paper we are interested in obtaining, with a joint probability or confidence coefficient $\geqslant 0{\cdot}95$, say, simultaneous confidence bounds on (i) a parametric function which is a measure of departure from H_0 (on all three variates), (ii) three parametric functions which are measures of departure from H_0 (on any two variates), and (iii) three parametric functions which are measures of departure from H_0 (on any one variate). Going back to the middle term of the inequality (4·19), we observe that here both A and C are of the full permissible rank, and calculate out $[C(A'A)^{-1}C']^{-1}$ and obtain the following parametric functions (on which we shall put confidence bounds):

For (i)
$$c^{\frac{1}{2}}_{\max}\left[\sum_{j=1}^{6} n_j(\eta_{jk}-\eta_k)(\eta_{jk'}-\eta_{k'})/n\right]=\Theta \quad \text{(say)},$$

where
$$\eta_k=\sum_{j=1}^{6} n_j\eta_{jk}/n\,(k=1,2,3) \quad \text{and} \quad \left[\sum_{j=1}^{6} n_j(\eta_{jk}-\eta_k)(\eta_{jk'}-\eta_{k'})/n\right]$$

$(k,k'=1,2,3)$ stands for a matrix whose k,k' element is the expression within [ ]. Likewise for (ii)

$$c^{\frac{1}{2}}_{\max}\left[\sum_{j=1}^{6} n_j(\eta_{jk}-\eta_k)(\eta_{jk'}-\eta_{k'})/n\right]$$
$$=\Theta^{(1)} \text{ (say)} \quad \text{when } k,k'=2,3,$$
$$=\Theta^{(2)} \text{ (say)} \quad \text{when } k,k'=1,3,$$

and
$$=\Theta^{(3)} \text{ (say)} \quad \text{when } k,k'=1,2.$$

Finally, for (iii) $c^{\frac{1}{2}}_{\max}\left[\sum_{j=1}^{6} n_j(\eta_{jk}-\eta_k)^2/n\right]$

$= \Theta^{(2,3)}$ (say) when $k = 1$,

$= \Theta^{(1,3)}$ (say) when $k = 2$,

and $= \Theta^{(1,2)}$ (say) when $k = 3$.

Notice that for (i) we have to deal with a 3×3 matrix, for (ii), three 2×2 matrices, and for (iii), three scalars for which $c^{\frac{1}{2}}_{\max}[\ \ ] = [\ \ ]^{\frac{1}{2}}$ itself. Notice also that where the null hypotheses are not true, then on the six groups, Θ is the positive square root of the characteristic root of the *population* 'between' product moment matrix for all the three variates, next $\Theta^{(1)}$, $\Theta^{(2)}$ and $\Theta^{(3)}$ are respectively the positive square roots of the characteristic roots of the *population* 'between' product moment matrices for the variates (2, 3), (1, 3) and (1, 2), and finally $\Theta^{(2,3)}$, $\Theta^{(1,3)}$ and $\Theta^{(1,2)}$ are respectively the positive square roots of the *population*

Table 1·1

(0)	(1)		(2)		(3)		(4)	
	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)
Matrix M $c^{\frac{1}{2}}_{\max}(M)$	sS^{**} 44·40	S^{*} 12·93	$sS^{**(1)}$ 41·42	$S^{*(1)}$ 12·77	$sS^{**(2)}$ 43·11	$S^{*(2)}$ 12·78	$sS^{**(3)}$ 28·65	$S^{*(3)}$ 9·81

(0)	(5)		(6)		(7)	
	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)
Matrix M $c^{\frac{1}{2}}_{\max}(M)$	$sS^{**(1,2)}$ 40·16	$S^{*(1,2)}$ 12·52	$sS^{**(1,3)}$ 12·30	$S^{*(1,3)}$ 3·35	$sS^{**(2,3)}$ 27·42	$S^{*(2,3)}$ 9·78

'between' sum of squares for the variates 1, 2 and 3. Up to this point the data and the analysis are entirely realistic. To set up the confidence bounds (4·19) and similar ones obtained by truncating the variates by ones, next by twos, we need c_α, or, in other words, the 5 % points of the relevant statistic with D.F. $p = 3$, $s = 5$ and $n-r = 134$. The construction of such tables is under way and the percentage points are not available to us at the moment. From certain inequality relations discussed in (Roy, 1953*a*) we set very approximately

$$c_\alpha(p, s, n-r) = c_{0\cdot05}(3, 5, 134) = 2\cdot90, \quad \text{so that} \quad sc_\alpha(\ \ ) = 14\cdot5 \quad \text{and} \quad [sc_\alpha]^{\frac{1}{2}} = 3\cdot81.$$

In Table 1·2, the 1st column gives confidence bounds on Θ, the 2nd, 3rd and 4th on $\Theta^{(1)}$, $\Theta^{(2)}$ and $\Theta^{(3)}$, respectively, and finally the 5th, 6th, and 7th on $\Theta^{(1,2)}$, $\Theta^{(1,3)}$ and $\Theta^{(2,3)}$, respectively. The Θ's being intrinsically non-negative, negative lower bounds would merely

imply that the corresponding Θ's could come down to zero values. The joint probability or confidence coefficient $\geqslant 0{\cdot}95$. As stated just now, the analysis and the conclusions are rendered somewhat approximate for lack of tables of percentage points, which, if available, would have made the confidence statements entirely correct.

It may be asked at this stage, what happens if we use Table 1·2 for testing of hypothesis. We observe that the procedure on testing of hypothesis, while related to, is not entirely identical with the procedure on confidence bounds. As discussed in (Roy, 1953) there is a procedure (which would be the right one) for testing H_0 with an *exact* preassigned probability, and, at the 5% level, H_0 would be just rejected on that level, but would not be rejected on a level, say, a little less than 5%. This agrees broadly with the conclusion given

Table 1·2

Variates cut out / Bounds	None	1st	2nd	3rd	1st and 2nd	1st and 3rd	2nd and 3rd
Lower bounds $[=(a)-[sc_\alpha]^{\frac{1}{2}}(b)]$	−4·86	−7·23	−5·58	−8·73	−7·54	−0·46	−9·84
Upper bounds $[=(a)+[sc_\alpha]^{\frac{1}{2}}(b)]$	93·66	90·07	91·80	66·03	87·86	25·06	64·68

in (Rao, 1952). However, if we use the above Table 1·2 on confidence bounds, for testing of hypothesis at a level $\leqslant 5\%$ (and recall the approximation introduced by taking a rough value of the percentage point), then, since all the confidence intervals include zero (which would correspond to the null hypothesis), we would say that we should just about not reject the null hypothesis on all 3 variates and on variates (1, 2), (1, 3) and (2, 3) and also on variates (1) (2) and (3).

5·2. To illustrate §3 we construct an artificial example by taking over S^{**} and S^{*} from the previous example and putting

$$p = q = 3,\ n = 8 \text{ (sample size } - 1),\quad B(3\times 3)\,B'(3\times 3) = S^{**}(3\times 3),\ S_{1.2}(3\times 3) = S^{*}(3\times 3)$$

and

$$S_{22}(3\times 3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I(3), \text{ (say)}.$$

Thus all $c(S_{22}^{-1}) = 1$. We have also, very approximately, $\theta_\alpha(p, q, n) = \theta_{0\cdot05}(3, 3, 8) = 0{\cdot}67$, so that $\lambda^2 = 0{\cdot}67/0{\cdot}33 = 2{\cdot}03$. Denoting by $\beta(3\times 3)$ the unknown population regression matrix of the p set on the q set, by $\beta^{(1)}(2\times 3)$, $\beta^{(2)}(2\times 3)$ and $\beta^{(3)}(2\times 3)$, respectively, the unknown population regression matrices of (2, 3), (1, 3) and (1, 2) on the q set and by $\beta^{(2,3)}(1\times 3)$, $\beta^{(1,3)}(1\times 3)$ and $\beta^{(1,2)}(1\times 3)$, respectively, the unknown population regression vectors of variates (1), (2) and (3) on the q set, we are interested in obtaining, with a joint probability $\geqslant 0{\cdot}95$, confidence bounds on $c_{\max}^{\frac{1}{2}}(\beta\beta') = \Theta$ (say), on $c_{\max}^{\frac{1}{2}}(\beta^{(i)}\beta^{(i)\prime}) = \Theta^{(i)}$(say, with $i = 1, 2, 3$) and on $c_{\max}^{\frac{1}{2}}(\beta^{(i,j)}\beta^{(i,j)\prime}) = \Theta^{(i,j)}$ (say with $i \neq j = 1, 2, 3$).

Each of these Θ's is a proper measure of departure from a corresponding null hypothesis.

We now have

Table 2·1

(0)	(1)		(2)		(3)		(4)	
	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)
Matrix M	BB'	$S_{1\cdot 2}$	$B^{(1)}B^{(1)'}$	$S^{(1)}_{1\cdot 2}$	$B^{(2)}B^{(2)'}$	$S^{(2)}_{1\cdot 2}$	$B^{(3)}B^{(3)'}$	$S^{(3)}_{1\cdot 2}$
$c^2_{\max}(M)$	19·86	12·93	18·52	12·77	19·28	12·78	12·81	9·81

(0)	(5)		(6)		(7)	
	(*a*)	(*b*)	(*a*)	(*b*)	(*a*)	(*b*)
Matrix M	$B^{(1,2)} \times B^{(1,2)'}$	$S^{(1,2)}_{1\cdot 2}$	$B^{(1,3)} \times B^{(1,3)'}$	$S^{(1,3)}_{1\cdot 2}$	$B^{(2,3)} \times B^{(2,3)'}$	$S^{(2,3)}_{1\cdot 2}$
$c^2_{\max}(M)$	17·96	12·52	5·50	3·35	12·26	9·78

Table 2·2

Variates of the p set cut out / Bounds	None	1st	2nd	3rd	1st and 2nd	1st and 3rd	2nd and 3rd
Lower [$= (a) - \lambda(b)$]	1·50	0·39	1·13	−1·12	0·18	0·74	−1·63
Upper [$= (a) + \lambda(b)$]	38·22	36·66	37·43	26·74	35·74	10·26	26·15

In Table 2·2, the 1st column gives confidence bounds on Θ, the 2nd, 3rd and 4th, respectively, on $\Theta^{(1)}$, $\Theta^{(2)}$ and $\Theta^{(3)}$, and the 5th, 6th and 7th respectively, on $\Theta^{(1,2)}$, $\Theta^{(1,3)}$ and $\Theta^{(2,3)}$, all with a joint probability $\geqslant 0{\cdot}95$. As before, the Θ's being intrinsically non-negative, any negative lower bound would merely imply that the corresponding Θ could come down to zero. The conclusion would be rough on account of the broad approximation in the percentage point.

Turning now to the problem of testing of the null hypothesis we recall that it is possible to do this at an *exact* preassigned level (Roy, 1953*a*). As observed on the previous example, we note also that the procedure of testing of hypothesis, while related to, is not entirely identical with that on confidence bounds. However, if we do use Table 2·2 for testing of hypothesis, then we first recall all the approximations involved and also that since $c_{\max}(M) \geqslant c_{\max}$(truncated M). We have now, necessarily,

$$\Theta \geqslant \Theta^{(1)},\ \Theta^{(2)},\ \Theta^{(3)};\quad \Theta^{(1)} \geqslant \Theta^{(1,3)},\ \Theta^{(1,2)};\quad \Theta^{(2)} \geqslant \Theta^{(2,3)},\ \Theta^{(1,2)} \quad \text{and} \quad \Theta^{(3)} \geqslant \Theta^{(1,3)},\ \Theta^{(2,3)}. \tag{5·2·1}$$

We note next that the only intervals that include zero are the columns 4 and 7. Thus we should roughly conclude that (i) not all variates (1, 2, 3) are independent of the q set, (ii) none of the variates (3) or (2) is independent of the q set, while variate (1) is independent of the q set and (iii) not all variates of either (2, 3) or (1, 3) or (1, 2) are independent of the q set. The conclusion (i) follows from the 1st column, (ii) from columns (5), (6) and (7), while (iii) follows from columns (2), (3) and (4) and also the inequalities (5·2·1).

Concluding remarks. We hope to be able to illustrate much better and with realistic examples and tables the methods of §§ 1, 2, 3 and 4 when two-tailed tables needed for §§ 1 and 2 and one-tailed tables needed for §§ 3 and 4 become available, as we expect them to be, in the near future.

In conclusion, it is a great pleasure to thank the editor and the referees for their valuable suggestions for the improvement of the paper both in form and in content.

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TABLES FOR USE IN ESTIMATING THE NORMAL DISTRIBUTION FUNCTION BY NORMIT ANALYSIS

PART I. DESCRIPTION AND USE OF TABLES

PART II. COMPARISON BETWEEN MINIMUM NORMIT χ^2 ESTIMATE AND THE MAXIMUM LIKELIHOOD ESTIMATE

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Part I. Description and use of tables

The cumulative normal distribution function has been used extensively in bio-assay and in other experiments in which P, the probability of some all-or-none 'response' is a monotonic function, increasing or decreasing, of a quantity x which measures the potency of the agent producing the response. The function may be written as in (1), with some auxiliary quantities defined by (2) and (3)

$$\tilde{P}_i = 1 - \tilde{Q}_i = \frac{1}{\sqrt{(2\pi)}} \int_{-\infty}^{\tilde{X}_i} e^{-\frac{1}{2}u^2} du, \tag{1}$$

$$\tilde{Z}_i = \frac{1}{\sigma\sqrt{(2\pi)}} e^{-\frac{1}{2}(x_i-\mu)^2/\sigma^2}, \tag{2}$$

$$\text{Normit of } \tilde{P}_i = \tilde{X}_i = (x_i - \mu)/\sigma = \alpha + \beta x_i. \tag{3}$$

$\tilde{P}_i$, the probability of response at x_i is given by (1) in terms of the integral of the standardized normal function $N(0, 1)$. The normal frequency curve, $N(\mu, \sigma)$ with ordinate $\tilde{Z}_i$ given by (2) is sometimes taken to represent the distribution of hypothetical resistances of the individuals in the experimental population.* From (3), $\alpha = -\mu/\sigma, \beta = 1/\sigma$. In bio-assay problems, one is frequently interested in the value of the dosage x corresponding to a 50 % response; this is given by $x_{50} = -\alpha/\beta$.†

Assuming this model, and given a set of observations x_i, n_i, r_i, where n_i is the number of individuals 'exposed' at x_i, and r_i is the number responding out of the n_i, a method of estimating the parameters, which is part of a procedure named 'normit analysis', has been investigated by Berkson (1955*a*). The estimator provided by this method falls in the class R.B.A.N. of Neyman (1949) (regular best asymptotically normal) and therefore is asymptotically equivalent to the maximum likelihood estimator. For finite samples, at least in a wide variety of conditions corresponding to situations met in practice, the variance and mean square error of this estimator are smaller than those of the maximum likelihood estimator.‡

With $p_i = 1 - q_i = r_i/n_i$ representing the observed relative frequency of response at x_i, and X_i representing the normit corresponding to p_i (observed normit or normit of p_i), if (1) gives

* This, however, is not a necessary part of the model. There are cases known in which such variable resistances may not even exist and still (1) may pretty well describe the observed phenomena. The model consists simply in the hypothesis that (1) evaluates the relevant probabilities, without implying necessarily any particular mechanism underlying them.

† Not to be confused with x_i ($i = 50$).

‡ See Part II.

the true probability of response at x_i, the observed X_i plotted against x_i should fall along the straight line $\hat{X}_i = \alpha + \beta x_i$, subject to random variation of the X_i. The estimate of normit analysis is obtained by minimizing

$$\chi^2 \text{ (normit)} = \Sigma n_i w_i (X_i - \hat{X}_i)^2, \tag{4}$$

where $\hat{X}_i = \hat{\alpha} + \hat{\beta} x_i$ is the estimate of $\alpha + \beta x_i$, and $w_i = Z_i^2/(p_i q_i)$ with Z_i given by (2) with p_i replacing $\bar{P}_i$. Since (4) is asymptotically distributed as χ^2, it is called the 'normit χ^2', and the estimate of normit analysis is called the 'minimum normit χ^2 estimate'*.

The minimization of (4) is obtained by accomplishing a weighted least squares fit of a straight line to the observations X_i, the weight of X_i being $n_i w_i$. The estimates of α and β, symbolized $\hat{\alpha}$ and $\hat{\beta}$, are explicit functions of the observations x_i, X_i, and are achieved simply and directly without use of iterative procedures, from the following equations:

$$\hat{\beta} = \frac{\Sigma n_i w_i (X_i - \bar{X})(x_i - \bar{x})}{\Sigma n_i w_i (x_i - \bar{x})^2} = \frac{\Sigma n_i w_i X_i x_i - \Sigma n_i w_i X_i \Sigma n_i w_i x_i / \Sigma n_i w_i}{\Sigma n_i w_i x_i^2 - (\Sigma n_i w_i x_i)^2 / \Sigma n_i w_i}, \tag{5}$$

$$\hat{\alpha} = \bar{X} - \hat{\beta}\bar{x}, \tag{6}$$

$$\hat{x}_{50} = -\hat{\alpha}/\hat{\beta}, \tag{7}$$

where $$\bar{X} = \Sigma n_i w_i X_i / \Sigma n_i w_i, \quad \bar{x} = \Sigma n_i w_i x_i / \Sigma n_i w_i.$$

For large samples the following formulas may be used to provide 'internal estimates' of the standard errors of these estimates

$$s^2(\bar{X}) = 1/\Sigma n_i w_i, \tag{8}$$

$$s^2(\hat{\beta}) = 1/\Sigma n_i w_i (x_i - \bar{x})^2, \tag{9}$$

$$s^2(\hat{\alpha}) = s^2(\bar{X}) + \bar{x}^2 s^2(\hat{\beta}), \tag{10}$$

$$s^2(\hat{x}_{50}) = \frac{1}{\hat{\beta}^2}[s^2(\bar{X}) + (\hat{x}_{50} - \bar{x})^2 s^2(\hat{\beta})]. \tag{11}$$

When α only has to be estimated equation (6) gives this estimate on replacing $\hat{\beta}$ by its known value β.

The calculations require for each observation the unit weight w_i and also $w_i X_i$. These are provided in Tables 1 and 2 (pp. 414–19 below). For each value of n between 2 and 50 inclusive, Table 1 gives the required quantities, with r as argument. Tabling in this way makes it unnecessary to compute $p = r/n$ and also provides w and wX with greater precision than usually would be obtained from the calculated value of p. For $n > 50$ it is necessary first to calculate p, and to use Table 2, which gives w and wX with p as argument. For $p = 0$ and $p = 1$, X is negatively and positively infinite and wX is zero. In such cases a working value is used taking p as $1/(2n)$ and $1 - 1/(2n)$ for the case of p equal to 0 and 1, respectively.†

The values of w and wX were calculated using the *Statistical Tables* of Kelley (1938), carrying the eight decimal places of those tables in the arithmetic computations; they are given in Tables 1 and 2 to five decimal places with the last digit correct to ± 1. The tables were checked by recomputation of sections of them in a spot check, and then by calculation and inspection of first and second differences. An example of the calculation of the estimates is shown in Table 3.

* A generalization of this class of estimates has since been formulated as 'minimum transform χ^2 estimates' (Berkson, 1954).

† See Part II.

Table 3. *Example of calculations; data from Fisher & Yates* (1953)

x	n	r	w^*	wX^*	wx
3	8	0	0·25813	−0·39601	0·77439
4	8	0	0·25813	−0·39601	1·03252
5	8	2	0·53857	−0·36326	2·69285
6	8	3	0·61350	−0·19549	3·68100
7	8	3	0·61350	−0·19549	4·29450
8	8	7	0·38743	0·44569	3·09944
9	8	8	0·25813	0·39601	2·32317

$$\Sigma w = 2{\cdot}92739, \quad \Sigma wx = 17{\cdot}89787, \quad \Sigma wX = -0{\cdot}70456,$$

$$\bar{x} = \frac{\Sigma wx}{\Sigma w} = 6{\cdot}11393, \quad \bar{X} = \frac{\Sigma wX}{\Sigma w} = -0{\cdot}24068,$$

$$\Sigma wx^2 = 117{\cdot}76905 \qquad \Sigma wXx = -0{\cdot}00013$$

$$(\Sigma wx)^2/\Sigma w = 109{\cdot}42640 \qquad \Sigma wX\,\Sigma wx/\Sigma w = -4{\cdot}30763$$

$$\Sigma w(x-\bar{x})^2 = \text{Diff.} = 8{\cdot}34265 \qquad \Sigma w(X-\bar{X})\,(x-\bar{x}) = \text{Diff.} = -4{\cdot}30750$$

$$\hat{\beta} = \frac{\Sigma w(X-\bar{X})\,(x-\bar{x})}{\Sigma w(x-\bar{x})^2} = 0{\cdot}5163, \quad \hat{\alpha} = \bar{X} - \hat{\beta}\bar{x} = -3{\cdot}3973,$$

$$\hat{x}_{50} = -\hat{\alpha}/\hat{\beta} = 6{\cdot}5801.$$

$$s^2(\bar{X}) = \frac{1}{n\Sigma w} = 0{\cdot}02470 \qquad (s(\bar{X}) = 0{\cdot}16),$$

$$s^2(\hat{\beta}) = \frac{1}{n\Sigma w(x-\bar{x})^2} = 0{\cdot}01498 \qquad (s(\hat{\beta}) = 0{\cdot}12),$$

$$s^2(\hat{\alpha}) = s^2\bar{X} + \bar{x}^2 s^2(\hat{\beta}) = 0{\cdot}6027 \qquad (s(\hat{\alpha}) = 0{\cdot}78),$$

$$s^2(\hat{x}_{50}) = \frac{1}{\hat{\beta}^2}[s^2(\bar{X}) + (\hat{x}_{50}-\bar{x})^2 s^2(\hat{\beta})] = 0{\cdot}1724 \qquad (s(\hat{x}_{50}) = 0{\cdot}42),$$

$$\hat{X} = 0{\cdot}52x - 3{\cdot}40.\dagger$$

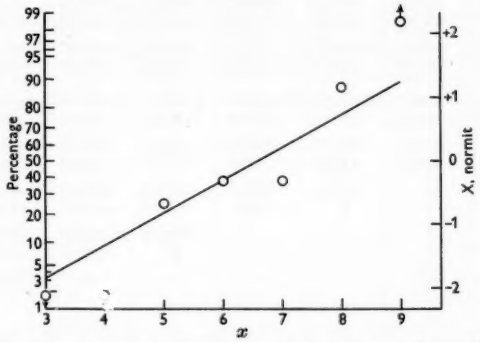

Fig. 1. The data of example for which calculations are shown in Table 3, together with the line fitted by minimum normit χ^2.

* From Table 1. If the n_i are not all equal as they are in the present example, these columns contain nw and nwX instead of w and wX. In case $n_i > 50$, one must first compute $p_i = r_i/n_i$, and consult Table 2.

† In practical work, it is useful to plot the data and the estimated normit line. A number of graph papers have been published for plotting linearly the cumulative normal distribution function. A convenient one for practical work, in respect to size and scale, is graph paper no. 32451, Normal ruling, Codex Book Company, Norwood, Mass., U.S.A. The present data are shown plotted on this graph paper in Fig. 1.

Table 1. *Normit weights*

Upper figure in table is $w = Z^2/pq$; lower figure is wX

For $r < \frac{1}{2}n$, wX is negative. For $r > \frac{1}{2}n$, use $n-r$ as argument, and wX is positive

r \ n		2	3	4	5	6	7	8	9	10
0	—	0·53857 ·36326	0·44951 ·43480	0·38743 ·44569	0·34222 ·43858	0·30763 ·42551	0·28031 ·41078	0·25813 ·39601	0·23974 ·38187	0·22394 ·36834
1	—	·63662 0	·59490 ·25630	·53857 ·36326	·48987 ·41228	·44951 ·43480	·41588 ·44390	·38743 ·44569	·36317 ·44332	·34222 ·43858
2	—	—	—	·63662 0	·62192 ·15756	·59490 ·25630	·56612 ·32042	·53857 ·36326	·51309 ·39240	·48987 ·41228
3	—	—	—	—	—	·63662 0	·62917 ·11321	·61350 ·19549	·59490 ·25630	·57567 ·30188
4	—	—	—	—	—	—	—	·63662 0	·63211 ·08838	·62192 ·15756
5	—	—	—	—	—	—	—	—	—	0·63662 0

r \ n	11	12	13	14	15	16	17	18	19	20
0	0·21059 ·35592	0·19881 ·34420	0·18850 ·33335	0·17916 ·32302	0·17090 ·31348	0·16345 ·30458	0·15689 ·29648	0·15093 ·28890	0·14520 ·28143	0·14014 ·27466
1	·32386 ·43243	·30763 ·42551	·29324 ·41823	·28031 ·41078	·26880 ·40342	·25813 ·39601	·24841 ·38875	·23974 ·38187	·23138 ·37487	·22394 ·36834
2	·46870 ·42582	·44951 ·43480	·43186 ·44062	·41588 ·44390	·40099 ·44547	·38743 ·44569	·37477 ·44488	·36317 ·44332	·35241 ·44119	·34222 ·43858
3	·55672 ·33663	·53857 ·36326	·52138 ·38385	·50513 ·39986	·48987 ·41228	·47555 ·42188	·46214 ·42923	·44951 ·43480	·43758 ·43895	·42638 ·44191
4	·60899 ·21245	·59490 ·25630	·58048 ·29162	·56612 ·32042	·55214 ·34389	·53857 ·36326	·52557 ·37920	·51309 ·39240	·50119 ·40330	·48987 ·41228
5	0·63360 ·07242	0·62645 ·13177	0·61697 ·18103	0·60623 ·22202	0·59490 ·25630	0·58337 ·28514	0·57182 ·30961	0·56049 ·33034	0·54940 ·34805	0·53857 ·36326
6	—	·63662 0	·63446 ·06132	·62917 ·11321	·62192 ·15756	·61350 ·19549	·60437 ·22815	·59490 ·25630	·58530 ·28064	·57567 ·30188
7	—	—	—	·63662 0	·63501 ·05307	·63092 ·09925	·62521 ·13937	·61843 ·17451	·61094 ·20533	·60306 ·23237
8	—	—	—	—	—	·63662 0	·63536 ·04687	·63211 ·08838	·62751 ·12492	·62192 ·15756
9	—	—	—	—	—	—	—	·63662 0	·63561 ·04193	·63298 ·07954
10	—	—	—	—	—	—	—	—	—	0·63662 0

Table 1 (*cont.*)

Upper figure in table is $w = Z^2/pq$; lower figure is wX

For $r < \frac{1}{2}n$, wX is negative. For $r > \frac{1}{2}n$, use $n-r$ as argument, and wX is positive

r \ n	21	22	23	24	25	26	27	28	29	30
0	0·13537	0·13091	0·12679	0·12301	0·11961	0·11615	0·11308	0·11042	0·10727	0·10499
	·26815	·26194	·25609	·25064	·24564	·24050	·23587	·23179	·22692	·22335
1	·21689	·21059	·20445	·19881	·19338	·18850	·18354	·17916	·17506	·17090
	·36190	·35592	·34990	·34420	·33855	·33335	·32791	·32302	·31833	·31348
2	·33268	·32386	·31564	·30763	·30029	·29324	·28673	·28031	·27449	·26880
	·43561	·43243	·42910	·42551	·42193	·41821	·41458	·41078	·40714	·40342
3	·41588	·40589	·39633	·38743	·37894	·37090	·36317	·35579	·34880	·34222
	·44390	·44511	·44567	·44569	·44525	·44445	·44332	·44193	·44032	·43858
4	·47907	·46870	·45885	·44951	·44048	·43186	·42361	·41588	·40822	·40099
	·41969	·42582	·43098	·43480	·43804	·44062	·44251	·44390	·44488	·44547
5	0·52812	0·51805	0·50829	0·49887	0·48987	0·48116	0·47281	0·46476	0·45693	0·44951
	·37625	·38738	·39698	·40525	·41228	·41834	·42350	·42791	·43168	·43480
6	·56612	·55672	·54757	·53857	·52984	·52138	·51309	·50513	·49738	·48987
	·32042	·33663	·35076	·36326	·37422	·38385	·39240	·39986	·40647	·41228
7	·59490	·58669	·57838	·57022	·56208	·55406	·54628	·53857	·53109	·52372
	·25630	·27735	·29617	·31272	·32760	·34088	·35263	·36326	·37272	·38128
8	·61570	·60899	·60204	·59490	·58771	·58048	·57327	·56612	·55909	·55214
	·18647	·21245	·23556	·25630	·27487	·29162	·30674	·32042	·33270	·34389
9	·62917	·62450	·61921	·61350	·60748	·60129	·59490	·58850	·58206	·57567
	·11321	·14355	·17086	·19549	·21775	·23787	·25630	·27294	·28812	·30188
10	0·63580	0·63360	0·63041	0·62645	0·62192	0·61697	0·61173	0·60623	0·60062	0·59490
	·03795	·07242	·10349	·13177	·15756	·18103	·20236	·22202	·23989	·25630
11	—	·63662	·63593	·63409	·63137	·62797	·62403	·61973	·61509	·61026
		0	·03461	·06640	·09532	·12181	·14617	·16842	·18903	·20786
12	—	—	—	·63662	·63604	·63446	·63211	·62917	·62573	·62192
				0	·03190	·06132	·08838	·11321	·13627	·15756
13	—	—	—	—	—	·63662	·63612	·63476	·63272	·63012
						0	·02951	·05688	·08223	·10585
14	—	—	—	—	—	—	—	·63662	·63619	·63501
								0	·02744	·05307
15	—	—	—	—	—	—	—	—	—	0·63662
										0

Table 1 (*cont.*)

Upper figure in table is $w = Z^2/pq$; lower figure is wX

For $r < \frac{1}{2}n$, wX is negative. For $r > \frac{1}{2}n$, use $n-r$ as argument, and wX is positive

r \ n	31	32	33	34	35	36	37	38	39	40
0	0·10223	0·09990	0·09802	0·09564	0·09371	0·09177	0·08981	0·08833	0·08634	0·08483
	·21897	·21523	·21219	·20830	·20514	·20191	·19862	·19612	·19272	·19013
1	·16737	·16381	·16019	·15689	·15393	·15093	·14789	·14520	·14249	·14014
	·30931	·30501	·30058	·29648	·29274	·28890	·28496	·28143	·27782	·27466
2	·26326	·25813	·25318	·24841	·24384	·23974	·23559	·23138	·22768	·22394
	·39964	·39601	·39237	·38875	·38517	·38187	·37844	·37487	·37166	·36834
3	·33589	·32984	·32386	·31820	·31285	·30763	·30276	·29781	·29324	·28884
	·43666	·43463	·43243	·43018	·42789	·42551	·42317	·42066	·41823	·41579
4	·39405	·38743	·38100	·37477	·36894	·36317	·35765	·35241	·34726	·34222
	·44572	·44569	·44540	·44488	·44419	·44332	·44231	·44119	·43994	·43857
5	0·44226	0·43535	0·42854	0·42214	0·41588	0·40977	0·40384	0·39827	0·39274	0·38743
	·43745	·43961	·44141	·44280	·44390	·44472	·44528	·44560	·44574	·44569
6	·48254	·47555	·46870	·46214	·45565	·44951	·44348	·43758	·43186	·42638
	·41743	·42188	·42582	·42923	·43225	·43480	·43703	·43895	·44062	·44191
7	·51660	·50970	·50286	·49631	·48987	·48357	·47755	·47161	·46587	·46025
	·38887	·39566	·40186	·40733	·41228	·41674	·42065	·42420	·42733	·43014
8	·54531	·53857	·53198	·52557	·51929	·51309	·50708	·50119	·49544	·48987
	·35402	·36326	·37164	·37920	·38608	·39240	·39810	·40330	·40802	·41228
9	·56928	·56301	·55672	·55058	·54449	·53857	·53268	·52694	·52138	·51583
	·31453	·32597	·33663	·34627	·35518	·36326	·37077	·37762	·38385	·38966
10	0·58917	0·58337	0·57757	0·57182	0·56612	0·56049	0·55490	0·54940	0·54392	0·53857
	·27128	·28514	·29791	·30961	·32042	·33034	·33955	·34805	·35599	·36326
11	·60522	·60014	·59490	·58966	·58443	·57919	·57393	·56873	·56358	·55843
	·22538	·24133	·25630	·27003	·28269	·29444	·30543	·31556	·32497	·33381
12	·61782	·61350	·60899	·60437	·59971	·59490	·59011	·58530	·58048	·57567
	·17724	·19549	·21245	·22815	·24262	·25630	·26892	·28064	·29162	·30188
13	·62711	·62370	·62005	·61620	·61212	·60795	·60370	·59932	·59490	·59049
	·12757	·14802	·16690	·18436	·20087	·21614	·23034	·24377	·25630	·26794
14	·63321	·63092	·62821	·62521	·62192	·61843	·61476	·61094	·60706	·60306
	·07701	·09925	·12009	·13937	·15756	·17451	·19038	·20533	·21923	·23237
15	0·63624	0·63520	0·63360	0·63157	0·62917	0·62645	0·62348	0·62030	0·61697	0·61350
	·02568	·04973	·07242	·09343	·11321	·13177	·14926	·16568	·18103	·19549
16	—	·63662	·63628	·63536	·63394	·63211	·62994	·62751	·62482	·62192
		0	·02425	·04687	·06830	·08838	·10726	·12492	·14169	·15756
17	—	—	—	·63662	·63632	·63550	·63423	·63259	·63062	·62839
				0	·02281	·04432	·06450	·08365	·10175	·11884
18	—	—	—	—	—	·63662	·63635	·63561	·63446	·63298
						0	·02154	·04193	·06132	·07954
19	—	—	—	—	—	—	—	·63662	·63638	·63571
								0	·02042	·03986
20	—	—	—	—	—	—	—	—	—	0·63662
										0

Table 1 (*cont.*)

Upper figure in table is $w = Z^2/pq$; lower figure is wX
For $r < \frac{1}{2}n$, wX is negative. For $r > \frac{1}{2}n$, use $n-r$ as argument, and wX is positive

r \ n	41	42	43	44	45	46	47	48	49	50
0	0·08330	0·08177	0·08022	0·07919	0·07762	0·07657	0·07498	0·07391	0·07283	0·07175
	·18750	·18483	·18212	·18029	·17751	·17563	·17277	·17084	·16889	·16692
1	·13776	·13537	·13335	·13091	·12886	·12679	·12512	·12301	·12132	·11961
	·27144	·26815	·26536	·26194	·25904	·25609	·25369	·25064	·24816	·24564
2	·22044	·21689	·21361	·21059	·20723	·20445	·20164	·19881	·19595	·19338
	·36517	·36190	·35881	·35592	·35265	·34990	·34709	·34420	·34124	·33855
3	·28460	·28031	·27644	·27252	·26880	·26503	·26148	·25813	·25475	·25160
	·41335	·41078	·40838	·40587	·40342	·40087	·39839	·39601	·39354	·39118
4	·33749	·33268	·32820	·32386	·31968	·31564	·31155	·30763	·30387	·30029
	·43716	·43561	·43405	·43243	·43078	·42910	·42731	·42551	·42371	·42193
5	0·38237	0·37721	0·37249	0·36769	0·36317	0·35876	0·35448	0·35032	0·34611	0·34222
	·44548	·44511	·44464	·44402	·44332	·44252	·44165	·44070	·43964	·43857
6	·42096	·41588	·41070	·40589	·40099	·39633	·39192	·38743	·38305	·37894
	·44303	·44390	·44461	·44511	·44547	·44567	·44574	·44569	·44551	·44525
7	·45474	·44951	·44429	·43924	·43437	·42955	·42478	·42022	·41588	·41147
	·43264	·43480	·43675	·43844	·43989	·44116	·44226	·44317	·44390	·44451
8	·48437	·47907	·47376	·46870	·46376	·45885	·45409	·44951	·44497	·44048
	·41619	·41969	·42294	·42582	·42842	·43080	·43292	·43480	·43651	·43804
9	·51041	·50513	·49993	·49480	·48987	·48505	·48023	·47555	·47100	·46649
	·39499	·39986	·40437	·40853	·41228	·41572	·41895	·42188	·42454	·42701
10	0·53330	0·52812	0·52307	0·51805	0·51309	0·50829	0·50359	0·49887	0·49436	0·48987
	·37001	·37625	·38200	·38738	·39240	·39698	·40123	·40525	·40887	·41228
11	·55337	·54837	·54342	·53857	·53374	·52903	·52437	·51996	·51534	·51091
	·34196	·34958	·35668	·36326	·36946	·37519	·38055	·38554	·39016	·39452
12	·57089	·56612	·56143	·55672	·55214	·54757	·54301	·53857	·53417	·52984
	·31143	·32042	·32873	·33663	·34389	·35076	·35725	·36326	·36892	·37422
13	·58606	·58157	·57713	·57275	·56819	·56394	·55961	·55528	·55105	·54684
	·27886	·28920	·29884	·30779	·31626	·32434	·33183	·33894	·34556	·35181
14	·59903	·59490	·59082	·58669	·58254	·57838	·57431	·57022	·56612	·56208
	·24463	·25630	·26711	·27735	·28704	·29617	·30465	·31272	·32042	·32760
15	0·60994	0·60623	0·60250	0·59873	0·59490	0·59109	0·58720	0·58337	0·57950	0·57567
	·20905	·22202	·23411	·24549	·25630	·26641	·27611	·28514	·29377	·30188
16	·61885	·61570	·61239	·60899	·60558	·60204	·59849	·59490	·59131	·58771
	·17254	·18647	·19982	·21245	·22421	·23556	·24620	·25630	·26585	·27487
17	·62593	·62331	·62049	·61759	·61454	·61142	·60820	·60496	·60162	·59829
	·13503	·15018	·16476	·17831	·19128	·20355	·21525	·22626	·23686	·24677
18	·63119	·62917	·62692	·62450	·62192	·61921	·61641	·61350	·61050	·60748
	·09689	·11321	·12882	·14355	·15756	·17086	·18345	·19549	·20697	·21775
19	·63467	·63332	·63169	·62982	·62776	·62552	·62317	·62064	·61806	·61534
	·05831	·07574	·09232	·10820	·12321	·13751	·15095	·16399	·17618	·18798
20	0·63640	0·63580	0·63485	0·63360	0·63211	0·63041	0·62850	0·62645	0·62425	0·62192
	·01946	·03795	·05561	·07242	·08838	·10349	·11806	·13177	·14494	·15756
21	—	·63662	·63642	·63587	·63501	·63386	·63249	·63092	·62917	·62725
		0	·01851	·03620	·05307	·06925	·08460	·09925	·11321	·12664
22	—	—	—	·63662	·63644	·63593	·63514	·63409	·63283	·63137
				0	·01771	·03461	·05084	·06640	·08112	·09532
23	—	—	—	—	—	·63662	·63646	·63599	·63526	·63429
						0	·01691	·03317	·04877	·06370
24	—	—	—	—	—	—	—	·63662	·63647	·63604
								0	·01627	·03190
25	—	—	—	—	—	—	—	—	—	0·63662
										0

Table 2. *Normit weights*

Upper figure in table is $w = Z^2/pq$; lower figure is wX
For $p < 0{\cdot}50$ on left, wX is negative. For $p > 0{\cdot}50$ on right, wX is positive

p	Thousandths, for p on left										
	0	1	2	3	4	5	6	7	8	9	
0·00	—	0·01135	0·02014	0·02799	0·03523	0·04203	0·04847	0·05463	0·06054	0·06624	0·07175	0·99
	—	·03507	·05796	·07690	·09343	·10825	·12177	·13424	·14584	·15670	·16692	
·01	0·07175	·07709	·08228	·08733	·09226	·09707	·10177	·10636	·11087	·11528	·11961	·98
	·16692	·17657	·18572	·19443	·20272	·21064	·21823	·22550	·23248	·23919	·24564	
·02	·11961	·12386	·12803	·13213	·13617	·14014	·14404	·14789	·15168	·15541	·15910	·97
	·24564	·25187	·25787	·26366	·26925	·27466	·27989	·28496	·28987	·29462	·29923	
·03	·15910	·16273	·16631	·16984	·17333	·17678	·18018	·18354	·18686	·19014	·19338	·96
	·29923	·30370	·30804	·31225	·31633	·32031	·32416	·32791	·33156	·33511	·33855	
·04	·19338	·19659	·19976	·20289	·20600	·20906	·21210	·21510	·21808	·22102	·22394	·95
	·33855	·34191	·34517	·34835	·35144	·35445	·35738	·36023	·36300	·36571	·36834	
0·05	0·22394	0·22682	0·22968	0·23251	0·23531	0·23809	0·24084	0·24357	0·24627	0·24895	0·25160	0·94
	·36834	·37091	·37340	·37584	·37821	·38051	·38276	·38495	·38708	·38916	·39118	
·06	·25160	·25423	·25684	·25942	·26199	·26453	·26705	·26955	·27203	·27449	·27693	·93
	·39118	·39315	·39507	·39694	·39875	·40052	·40225	·40392	·40555	·40714	·40868	
·07	·27693	·27935	·28175	·28413	·28649	·28884	·29116	·29347	·29576	·29804	·30029	·92
	·40868	·41019	·41165	·41307	·41445	·41579	·41709	·41836	·41958	·42078	·42193	
·08	·30029	·30253	·30476	·30697	·30916	·31134	·31350	·31564	·31777	·31989	·32199	·91
	·42193	·42306	·42415	·42520	·42622	·42722	·42817	·42910	·43000	·43087	·43171	
·09	·32199	·32407	·32614	·32820	·33024	·33227	·33429	·33629	·33828	·34026	·34222	·90
	·43171	·43251	·43330	·43405	·43477	·43547	·43614	·43679	·43741	·43800	·43857	
0·10	0·34222	0·34417	0·34611	0·34803	0·34994	0·35184	0·35373	0·35560	0·35747	0·35932	0·36116	0·89
	·43857	·43912	·43964	·44013	·44061	·44106	·44148	·44189	·44227	·44263	·44297	
·11	·36116	·36299	·36480	·36661	·36840	·37019	·37196	·37372	·37547	·37721	·37894	·88
	·44297	·44329	·44359	·44386	·44412	·44436	·44457	·44477	·44495	·44511	·44525	
·12	·37894	·38066	·38237	·38407	·38576	·38743	·38910	·39076	·39241	·39405	·39568	·87
	·44525	·44537	·44548	·44556	·44563	·44569	·44572	·44574	·44574	·44572	·44569	
·13	·39568	·39730	·39891	·40051	·40210	·40369	·40526	·40682	·40838	·40993	·41147	·86
	·44569	·44564	·44558	·44550	·44540	·44529	·44517	·44502	·44487	·44470	·44451	
·14	·41147	·41299	·41452	·41603	·41753	·41903	·42051	·42199	·42346	·42492	·42638	·85
	·44451	·44431	·44410	·44387	·44363	·44338	·44311	·44283	·44254	·44223	·44191	
·15	0·42638	0·42782	0·42926	0·43069	0·43211	0·43353	0·43493	0·43633	0·43772	0·43910	0·44048	0·84
	·44191	·44158	·44123	·44088	·44051	·44012	·43973	·43933	·43891	·43848	·43804	
·16	·44048	·44185	·44321	·44456	·44591	·44725	·44858	·44990	·45122	·45253	·45383	·83
	·43804	·43759	·43713	·43665	·43617	·43567	·43516	·43465	·43412	·43358	·43303	
·17	·45383	·45513	·45642	·45770	·45898	·46025	·46151	·46276	·46401	·46525	·46649	·82
	·43303	·43247	·43191	·43133	·43074	·43014	·42953	·42892	·42829	·42765	·42701	
·18	·46649	·46772	·46894	·47016	·47136	·47257	·47376	·47495	·47614	·47732	·47849	·81
	·42701	·42635	·42569	·42502	·42433	·42364	·42294	·42224	·42152	·42080	·42006	
·19	·47849	·47965	·48081	·48196	·48311	·48425	·48539	·48652	·48764	·48876	·48987	·80
	·42006	·41932	·41857	·41781	·41705	·41627	·41549	·41470	·41390	·41310	·41228	
·20	0·48987	0·49097	0·49207	0·49317	0·49425	0·49534	0·49641	0·49748	0·49855	0·49961	0·50066	0·79
	·41228	·41146	·41063	·40980	·40895	·40810	·40725	·40638	·40551	·40463	·40375	
·21	·50066	·50171	·50275	·50379	·50482	·50585	·50687	·50789	·50890	·50991	·51091	·78
	·40375	·40285	·40195	·40105	·40013	·39921	·39829	·39735	·39642	·39547	·39452	
·22	·51091	·51190	·51289	·51387	·51485	·51583	·51680	·51776	·51872	·51967	·52062	·77
	·39452	·39356	·39259	·39162	·39065	·38966	·38868	·38768	·38668	·38567	·38466	
·23	·52062	·52157	·52250	·52344	·52437	·52529	·52621	·52712	·52803	·52894	·52984	·76
	·38466	·38364	·38262	·38159	·38055	·37951	·37846	·37741	·37636	·37529	·37422	
·24	·52984	·53073	·53162	·53251	·53339	·53426	·53513	·53600	·53686	·53772	·53857	·75
	·37422	·37315	·37207	·37099	·36990	·36881	·36771	·36660	·36549	·36438	·36326	
		9	8	7	6	5	4	3	2	1	0	p
		Thousandths, for p on right										

Table 2 (*cont.*)

Upper figure in table is $w = Z^2/pq$; lower figure is wX
For $p < 0{\cdot}50$ on left, wX is negative. For $p > 0{\cdot}50$ on right, wX is positive

p	Thousandths, for p on left											
	0	1	2	3	4	5	6	7	8	9		
0·25	0·53857 ·36326	0·53942 ·36214	0·54026 ·36101	0·54110 ·35987	0·54193 ·35874	0·54276 ·35759	0·54359 ·35645	0·54441 ·35529	0·54523 ·35414	0·54604 ·35298	0·54684 ·35181	0·74
·26	·54684 ·35181	·54765 ·35064	·54845 ·34946	·54924 ·34829	·55003 ·34710	·55081 ·34591	·55159 ·34472	·55237 ·34353	·55314 ·34233	·55391 ·34112	·55468 ·33991	·73
·27	·55468 ·33991	·55543 ·33870	·55619 ·33748	·55694 ·33626	·55769 ·33504	·55843 ·33381	·55917 ·33257	·55990 ·33134	·56063 ·33010	·56136 ·32885	·56208 ·32760	·72
·28	·56208 ·32760	·56280 ·32635	·56351 ·32510	·56422 ·32384	·56493 ·32257	·56563 ·32131	·56632 ·32003	·56702 ·31876	·56771 ·31748	·56839 ·31620	·56907 ·31492	·71
·29	·56907 ·31492	·56975 ·31363	·57042 ·31234	·57109 ·31104	·57176 ·30974	·57242 ·30844	·57308 ·30713	·57373 ·30583	·57438 ·30451	·57503 ·30320	·57567 ·30188	·70
0·30	0·57567 ·30188	0·57631 ·30056	0·57694 ·29923	0·57757 ·29791	0·57820 ·29657	0·57882 ·29524	0·57944 ·29390	0·58005 ·29256	0·58066 ·29122	0·58127 ·28987	0·58188 ·28852	0·69
·31	·58188 ·28852	·58248 ·28717	·58307 ·28582	·58366 ·28446	·58425 ·28310	·58484 ·28173	·58542 ·28037	·58600 ·27900	·58657 ·27762	·58714 ·27625	·58771 ·27487	·68
·32	·58771 ·27487	·58827 ·27349	·58883 ·27211	·58939 ·27072	·58994 ·26933	·59049 ·26794	·59103 ·26655	·59158 ·26515	·59211 ·26375	·59265 ·26235	·59318 ·26095	·67
·33	·59318 ·26095	·59371 ·25954	·59423 ·25813	·59475 ·25672	·59527 ·25531	·59578 ·25389	·59629 ·25247	·59679 ·25105	·59730 ·24963	·59780 ·24820	·59829 ·24677	·66
·34	·59829 ·24677	·59878 ·24534	·59927 ·24391	·59976 ·24248	·60024 ·24104	·60072 ·23960	·60119 ·23816	·60166 ·23671	·60213 ·23527	·60260 ·23382	·60306 ·23237	·65
0·35	0·60306 ·23237	0·60351 ·23092	0·60397 ·22946	0·60442 ·22801	0·60487 ·22655	0·60531 ·22509	0·60575 ·22363	0·60619 ·22216	0·60662 ·22070	0·60706 ·21923	0·60748 ·21776	0·64
·36	·60748 ·21776	·60791 ·21629	·60833 ·21481	·60875 ·21334	·60916 ·21186	·60957 ·21038	·60998 ·20890	·61038 ·20741	·61078 ·20593	·61118 ·20444	·61158 ·20295	·63
·37	·61158 ·20295	·61197 ·20146	·61236 ·19997	·61274 ·19848	·61312 ·19698	·61350 ·19549	·61387 ·19399	·61425 ·19249	·61462 ·19098	·61498 ·18948	·61534 ·18798	·62
·38	·61534 ·18798	·61570 ·18647	·61606 ·18496	·61641 ·18345	·61676 ·18194	·61711 ·18043	·61745 ·17891	·61779 ·17740	·61812 ·17588	·61846 ·17436	·61879 ·17284	·61
·39	·61879 ·17284	·61912 ·17132	·61944 ·16979	·61976 ·16827	·62008 ·16674	·62039 ·16522	·62070 ·16369	·62101 ·16216	·62132 ·16063	·62162 ·15910	·62192 ·15756	·60
0·40	0·62192 ·15756	0·62221 ·15603	0·62251 ·15449	0·62280 ·15295	0·62308 ·15141	0·62337 ·14987	0·62365 ·14833	0·62392 ·14679	0·62420 ·14525	0·62447 ·14370	0·62474 ·14216	0·59
·41	·62474 ·14216	·62500 ·14061	·62526 ·13906	·62552 ·13751	·62578 ·13596	·62603 ·13441	·62628 ·13286	·62653 ·13130	·62677 ·12975	·62701 ·12819	·62725 ·12664	·58
·42	·62725 ·12664	·62748 ·12508	·62772 ·12352	·62794 ·12196	·62817 ·12040	·62839 ·11884	·62861 ·11728	·62883 ·11572	·62904 ·11415	·62925 ·11259	·62946 ·11102	·57
·43	·62946 ·11102	·62966 ·10945	·62986 ·10789	·63006 ·10632	·63026 ·10475	·63045 ·10318	·63064 ·10161	·63082 ·10004	·63101 ·09846	·63119 ·09689	·63137 ·09532	·56
·44	·63137 ·09532	·63154 ·09374	·63171 ·09217	·63188 ·09059	·63205 ·08901	·63221 ·08744	·63237 ·08586	·63252 ·08428	·63268 ·08270	·63283 ·08112	·63298 ·07954	·55
0·45	0·63298 ·07954	0·63312 ·07796	0·63326 ·07638	0·63340 ·07480	0·63354 ·07321	0·63367 ·07163	0·63380 ·07005	0·63393 ·06846	0·63405 ·06688	0·63417 ·06529	0·63429 ·06370	0·54
·46	·63429 ·06370	·63441 ·06212	·63452 ·06053	·63463 ·05894	·63473 ·05736	·63484 ·05577	·63494 ·05418	·63503 ·05259	·63513 ·05100	·63522 ·04941	·63531 ·04782	·53
·47	·63531 ·04782	·63540 ·04623	·63548 ·04464	·63556 ·04305	·63564 ·04146	·63571 ·03986	·63578 ·03827	·63585 ·03668	·63592 ·03509	·63598 ·03349	·63604 ·03190	·52
·48	·63604 ·03190	·63609 ·03031	·63615 ·02871	·63620 ·02712	·63625 ·02552	·63629 ·02393	·63633 ·02234	·63637 ·02074	·63641 ·01915	·63644 ·01755	·63647 ·01596	·51
·49	·63647 ·01596	·63650 ·01436	·63653 ·01277	·63655 ·01117	·63657 ·00957	·63658 ·00798	·63660 ·00638	·63661 ·00479	·63661 ·00319	·63662 ·00160	·63662 ·00000	·50
	9	8	7	6	5	4	3	2	1	0		
	Thousandths, for p on right											p

Part II. Comparison between the minimum normit χ^2 estimate and the maximum likelihood estimate

1. *Description of the investigation*

The statement in Part I (p. 411), characterizing the minimum normit χ^2 estimate as having smaller variance and mean square error than the maximum likelihood estimate, was made in a preliminary version of the present paper, on the basis of results obtained in a planned programme of sampling experiments (Berkson, 1955*a*). The editor of *Biometrika*, in his comments on the manuscript, conveyed the suggestion that perhaps these experiments were not a sufficient support for so firm a statement. Originally I had thought that the statement was supported well enough. However, on reflexion, I realized that my own conviction was based, in addition to the specific sampling results cited, on cumulated auxiliary results gained in the course of investigations of similar questions going over many years. I decided that the editor's comment was sound, for it was not to be expected that an average reader, if he found the conclusions surprising, would sympathetically seek out all this evidence. And so I was prompted to initiate a new programme of calculations, the results of which are presented here. These results and those obtained in previous studies are summarized in the concluding section.

It was decided in the first place to calculate the bias and variance of the estimates by actual summation over the total sampling population for all the primary situations in which, previously, these statistics had been estimated with the use of stratified random samples. With the precise values of these statistics thus made available, it was possible to explore some other questions, for which even good approximations were inadequate, notably some questions concerned with the lower bounds of the variance and the mean square error. For similar reasons the distributions of the sample χ^2's were investigated. In addition, the bias and variance for an auxiliary experiment, corresponding to the estimate of relative potency, were calculated using a stratified random sample.

The situation dealt with simulates a bio-assay experiment with three equally spaced 'dosages' x_i, at each of which $n_i = 10$ animals are 'exposed'. The primary experiment was with dosages scaled $-1, 0, +1$ at which the true probability of response P_i was respectively 0·3, 0·5 and 0·7, which determined the parameters as $\alpha = 0$, $\beta =$ normit 0·7. For the same parameters and with the same equal spacing of the dosages, experiments were performed also with the probability of response at central x_i, $P_c = 0{\cdot}6$, 0·7 and 0·8. Since p_i, the proportion of animals responding at each x_i, may have any of 11 possible values, the total number of possible sets of responses is 1331. It was, therefore, necessary to obtain the estimates for each of 1331 possible samples. The programme of experiments was in two sections, one in which β was considered known, with only α to be estimated, the other in which both parameters were to be estimated simultaneously. In what follows, the procedures which were used are described in terms of the estimation of both parameters, from which the corresponding procedures used for the estimation of α alone should be clear.

The minimum normit χ^2 estimates of β and α have been given in equations (5) and (6) of Part I above, together with definitions of p_i, q_i, X_i, w_i. n_i, the number exposed to risk at x_i, is 10 for all values of i.

The values of X_i and Z_i required for evaluation of (5) and (6) were obtained as for Table 1,

and the estimates were written to four decimal places correct to ± 1 in the last place. Each estimate was checked by inserting the values in the estimating equations

$$\Sigma n_i w_i (X_i - \hat{X}_i) = 0, \tag{12}$$

$$\Sigma n_i w_i x_i (X_i - \hat{X}_i) = 0, \tag{13}$$

where $\hat{X}_i = \hat{\alpha} + \hat{\beta} x_i$. Check of the estimate was performed by a different computer than the one who calculated its value. Because of symmetry among the samples, not all 1331 of them required independent computation. For estimate of both α and β, only about a quarter, and for estimate of α alone, about half of the samples required independent calculation, the estimates for the uncomputed samples being obtainable from those computed, by appropriate change of sign.

Since for $p_i = 0$ or $p_i = 1$, the corresponding value of X_i is infinite, some rule must be adopted to modify these observations, if a finite estimate is to be obtained with samples containing either of them. I have used the rule of substituting for an observation of $p_i = 0$, a working value $p_i = 1/(2n_i)$ and for $p_i = 1$, a working value $p_i = 1 - 1/(2n_i)$.*

To obtain the maximum likelihood estimate an iterative procedure must be used. For the present experiment two methods were employed. For the most part the procedure utilizing the linear transform as outlined by Fisher & Yates (1953) was used, and with this the tables of Finney & Stevens (1948) were utilized; secondarily the method set forth by Garwood (1941) was employed, for which convenient tables have been provided by Cornfield & Mantel (1950). The two methods are fairly comparable in respect to the amount of arithmetical labour involved, and no regular difference was noted in regard to speed of convergence, but the linear transform method is more directly analogous to the common method of fitting a straight line by least squares, and therefore more familiar to the computers—this was the chief reason for preferring it. The estimates as used for calculation of the statistics were to be correct to better than ± 1 in the third decimal place and therefore were set down with four decimal places, the last being correct to ± 5. In order to attain estimates of the desired precision, the following procedure was used: with $\hat{\alpha}_0$ and $\hat{\beta}_0$ (provisional values of the estimates) decided on, $\partial\hat{\alpha}$ and $\partial\hat{\beta}$ (the respective corrections to these for improved estimates) were computed using the Finney–Stevens tables. If either $\partial\hat{\alpha}$ or $\partial\hat{\beta}$ was as large as 0·01, another iteration was performed in the same way, and this process was continued until both $\partial\hat{\alpha}$ and $\partial\hat{\beta}$ were less than 0·01. The argument of the Finney–Stevens tables is carried to two decimal places only and cannot conveniently be used beyond this stage of the iterations. From this point on, the computations were accomplished using the W.P.A. tables (Lowan,

* This is an old 'empirical' rule, the origin of which I do not know. Another rule has been advanced by Reed (1936), still another by Gaddum (1933), and doubtless there are others besides these which have been employed. Elsewhere I (Berkson, 1955*b*) have set forth some considerations on the basis of which the $2n$ rule may be considered reasonable, but I do not advance them as necessarily preferable to any other. It should be noted that the contrast between the minimum normit χ^2 estimate and the maximum likelihood estimate does not essentially consist in the different methods used for dealing with the observations of zero or 100 % response. In maximum likelihood estimation the rule used is the application, to the particular observations of zero or 100 % response, of a modification required to be applied to all observations (Fisher, 1954), whereas in minimum normit χ^2 estimation, except for the observations of zero and 100 %, there is no modification of the observations at all—the working value is simply the observation itself. And of course there is the difference in the weights used. In maximum likelihood estimation the weight for unit observation is $\hat{z}^2/\hat{p}\hat{q}$, where the capped quantities refer to the corresponding maximum likelihood estimates, whereas in the minimum normit χ^2 estimation the unit weight is Z^2/pq, where the quantities represented correspond to the observations directly.

1942), which are more detailed though less convenient than the Finney-Stevens tables. Iteration was continued until both $\partial\hat{\alpha}$ and $\partial\hat{\beta}$ were less than 0·0005. The estimates obtained with these adjustments, written to four decimal places,* were checked by insertion in the estimating equations

$$\Sigma(n_i\hat{Z}_i/\hat{P}_i\hat{Q}_i)(p_i-\hat{P}_i) = 0, \tag{14}$$

$$\Sigma(n_i x_i\hat{Z}_i/\hat{P}_i\hat{Q}_i)(p_i-\hat{P}_i) = 0, \tag{15}$$

where $\hat{P}_i = 1-\hat{Q}_i$ is the estimate of $\bar{P}_i$, the probability at x_i, given by the estimated parameters, $\hat{Z}_i$ is the corresponding normal ordinate and p_i is the sample value of the relative frequency. As in the case with the minimum normit χ^2 estimate, independent computation was necessary for only about half of the samples for the estimate of α alone, and for about a quarter of the samples for the estimate of both parameters.

The number of iterative cycles required depends of course on how good an approximation the provisional values $\hat{\alpha}_0$, $\hat{\beta}_0$ are of the maximum likelihood estimate. For the present experiment, the samples were ordered systematically to facilitate computation, each sample differing from the previous one in only a single value of the observed frequency. This enabled the computers to guess the values of the provisional estimates from the precise solution of the previous sample. Even with this arrangement the number of iterative cycles required varied considerably. In some regions the procedure required two cycles with the Finney–Stevens tables, followed by two with the W.P.A. tables, in others two with the Finney–Stevens tables, and only one with the W.P.A. tables. At most we used five cycles, while some samples required only two cycles. However, so much depends on the sample, and on how well the provisional values are estimated, that really, no good judgement can be made as to how many iterative cycles are required to obtain a satisfactory maximum likelihood estimate. Indeed, the point to stress is that one cannot depend on just how many cycles have been calculated. What one requires is some objective methodical scheme by which one can recognize whether another cycle is necessary.

Each estimate was punched into a Hollerith punch card, together with its sample for identification. By contract with a service bureau of the International Business Machines Corporation, the square of the estimate of each parameter and their product were calculated with the use of IBM calculating machine number 164 and checked with the same machine, and the several results were punched into the same card. Other multiplications required for the calculations were performed in the same way.

The 1331 cards thus prepared, each punched with its sample, the estimate $\hat{\alpha}$, the estimate $\hat{\beta}$, the square of each, and their product, constituted the basic instrument for the rest of the computations. Corresponding to each dosage arrangement for which statistics were to be computed, it was necessary now to calculate the probability of each sample. For instance, in the dosage arrangement with $P_c = 0{\cdot}5$, the probability P_i at the lowest dosage is 0·3 and the probability P_i at the highest dosage is 0·7. For each of the eleven possible samples at each dosage the probability was directly calculated, checked, and hand-punched into auxiliary

* In the corresponding experiment with the logistic function (Berkson, 1955*b*), the estimates were computed, correct to one more place than in the present experiment. However, this was really more than was necessary for the purposes in hand, and the normit function is considerably more difficult to compute than the logit function. Actually, essentially the same results would have been obtained, so far as concerns the calculated values of the mean square error, if even one less place were retained, but it was considered desirable that the estimates should be calculated as precisely as practicable with the available computing facilities.

cards. Each combination of three of these, one corresponding to each of the three dosages, was multiplied together in two steps with the use of the automatic multiplying punch machine to give the probability of each sample, and each multiplication was checked. The resulting 1331 probabilities were transferred to the corresponding main cards with the use of a reproducing punch machine preparatory to multiplying them by the estimates, their squares and products. These multiplications were accomplished with the use of the automatic multiplying punch, and the results were punched into the same main card and checked. The sums of estimates, sums of squares and sums of products necessary for calculation of the required statistics were now obtained by totalling these quantities with the use of an available IBM punch-card tabulator.

For dosage arrangement with probability of the central dosage $P_c = 0{\cdot}5$, the dosage values x_i at the successive dosages were taken as $-1, 0, +1$, and the values of $\hat{\alpha}$ and $\hat{\beta}$, the estimates of α and β, respectively, which were punched into the master cards, were entered as for this dosage arrangement. With other equally spaced dosage arrangements, for any specific sample, $\hat{\beta}$, the estimate of β, will be the same, but the estimate of α will be different and is given by

$$\hat{\alpha} = \hat{\alpha}_{0{\cdot}5} - C\hat{\beta}, \tag{16}$$

where $\hat{\alpha}_{0{\cdot}5}$ is the estimate for that sample with $P_c = 0{\cdot}5$, and $C = \beta^{-1} \times$ normit of P_c. Accordingly the estimates did not have to be computed anew for each new experimental set-up, since all the experiments dealt with were for three dosages, equally spaced. It was necessary, however, to calculate anew the probability of each sample, from the probabilities corresponding to the new dosage positions.* These were punched into the main cards, the values $\hat{\alpha}_{0{\cdot}5}$, $\hat{\alpha}^2_{0{\cdot}5}$, $\hat{\beta}$, $\hat{\beta}^2$, $\hat{\alpha}_{0{\cdot}5}\hat{\beta}$ having already been punched into them. The five multiplications of the sample probability by these quantities were accomplished, and the results were punched into the cards and checked as for the calculations with $P_c = 0{\cdot}5$. By totalling these, the required sums of estimates, the sums of squares and the sums of products are obtained, and from them the required statistics were computed for the new dosage arrangement, taking into account relation (16).

2. *Results*

The primary purpose of these experiments was to compare the error of the minimum normit χ^2 estimate with the error of the maximum likelihood estimate. An indefinite number of criteria for such a comparison are conceivable; I used the classic measure of error, the mean square error, that is, the expected value of the square of the difference of the estimate from the true value of the parameter.

This cannot be the place for an adequate discussion of such a fundamental idea as the concept of error. However, I should like to note that at the beginning of these investigations,

* It is of some interest to note that this is not the case for the equivalent situation with the logistic function. That function has simple sufficient statistics which for α and β are respectively Σr_i and $\Sigma r_i x_i$. Accordingly the relative frequency of samples, within sets of samples having the same value for these statistics, remains the same, independent of the change of parameters. The total probability of such a set, with a change of parameter, therefore can be obtained from the probability with the original value of the parameter, by multiplication with a determinable constant. A change of dosage arrangement is equivalent to a change of the parameter α, and for the experiments with α to be estimated, β being known, since there are only 31 sets of samples defined by Σr_i, the required probabilities with change of dosage arrangement can be determined by 31 simple multiplications instead of 1331 more involved multiplications. This facilitation illustrates the great general statistical advantage of working with a function that possesses simple sufficient statistics for estimating its parameters.

I did consider and make calculations on the basis of other criteria, as well as with the mean square error. In particular I explored the idea of comparing the discrepancy of the estimates from their parameters directly in terms of the distribution of the estimates. This is related to what Savage (1954) calls the 'principle of inadmissibility', and to what Pitman (1937) has investigated as the 'closeness' of an estimate. According to this principle, if T is an estimate of θ and e is the error $(T-\theta)$ of estimate, then if the probability of an error equal to or greater than $|e|$ is smaller for an estimator T_1 than for T_2 (excluding the values of e for which the probabilities are equal) for all values of e and for all values of θ, then T_1 is a better estimator than T_2 in an absolute sense. Intuitively this principle seems incapable of denial, and I am not disposed to question it here. However, I do not believe that it can be very useful in providing a criterion for comparison of estimates in situations which are met in practice. For it will only rarely occur that two seriously recommended estimates are comparable in this respect at all; it will not even be typical for one to be better than the other in this stringent sense at any given θ, let alone one being better than the other in this sense at all θ.

If for all values of θ, T_1 and T_2 are normally distributed and are both unbiased, and the variance of T_1 is less than the variance of T_2, then T_1 is better than T_2 in the absolute sense defined. So, for instance, the estimate of the mean of a normal distribution, from the mean of a sample based on $n_1 > n_2$, is better, in the absolute sense, than an estimate based on a sample of n_2. But this is a case with the estimate represented by a continuous variate. If we consider a discontinuous variate, for instance the estimate of a probability P, with the usual estimate $T = s/n$, where s is the number of observed events in a sample of n, the case is already not so simple. If for T_1, $n = 3$, and for T_2, $n = 2$, then both estimators are unbiased, and the variance of T_1 is less than that of T_2. But T_1 is not necessarily better than T_2 in the absolute sense defined, as can be seen if we consider the case with $P = 0{\cdot}5$. In this case the probability of T_1 yielding the correct value of the parameter is zero, while for T_2 it is 0·5, so that, for instance, the probability of an error equal to or greater than $|0{\cdot}1|$ is 1 for T_1 and 0·5 for T_2. In comparing the minimum normit χ^2 estimate with the maximum likelihood estimate in terms of the distribution of the estimates, I encountered just such 'paradoxes'. Although, philosophically the mean square error cannot be considered necessarily the only relevant measure of error, I hardly believe that a better criterion for general application will be found. This is to say, if for some practical situation, an estimator T_1 is found to have a smaller mean square error than T_2, for all reasonable values of the parameter, I doubt that T_2 will be found better than T_1 by some other reasonable criterion.

Both estimates are asymptotically efficient (Taylor, 1953*a*). This is interpreted to mean that with x_i's fixed, as the number n_i at each x_i is increased indefinitely, the estimate approaches a normal distribution the variance of which is equal to or less than that of any other similarly distributed estimate. It is to be noted that the properties refer to the never-attained limiting distribution and that mathematically asymptotic efficiency does not imply necessarily any attribute of the estimate with finite samples. However, it is reasonable to assume, for practical cases like the present one, that for large n_i at each x_i, both estimates will have about the same variance. In one experiment with three equally spaced doses, corresponding respectively to true $\bar{P}$'s 0·3, 0·5, 0·7, 50 at each dose, using an amply large stratified random sample, this was corroborated. With $\beta = 0{\cdot}524401$, considered known and $\alpha = 0$ to be estimated, the theoretical value of the asymptotic variance correct to six decimal places was 0·011186, the determined value of the variance for the minimum normit χ^2 estimate was 0·011186, that is the same, to that degree of precision, while that for the

maximum likelihood estimate was only slightly higher, namely 0·011191 (Berkson, 1955*a*). For both estimators, to four significant figures and five decimal places the variance had attained its asymptotic value.

With small samples, comparison of the error of the estimates is complicated by the fact that for some samples the maximum likelihood estimate is infinite. In each such case the sample was omitted in computing the statistics of the minimum normit χ^2 estimate as well as of the maximum likelihood estimate. With probability at central dosage $\tilde{P}_c = 0{\cdot}5$, either in the case of estimating only α or in that of estimating both parameters, the frequency of such samples in the sampling population is so very small that their omission can reasonably be considered not to affect the virtual comparison of the estimates. But as the dosage arrangement is changed from the symmetrical one with $\tilde{P}_c = 0{\cdot}5$, to asymmetrical arrangements with $\tilde{P}_c$ appreciably different from 0·5, the probability of such samples increases, and the calculation of the statistics omitting them becomes questionable. I have made comparisons only for situations in which the samples yielding an infinite estimate by maximum likelihood constitute less than 5 % of the sampling distribution. When both parameters are to be estimated, with $\tilde{P}_c = 0{\cdot}9$, 25·1 % of the samples yield infinite estimates by maximum likelihood. The upper limit of the experimental arrangements was therefore taken as the one corresponding to $\tilde{P}_c = 0{\cdot}8$, for which the probability of samples with an infinite maximum likelihood estimate is 4·4 %. In order to co-ordinate the computation, the experiments estimating only α were limited to this region also. Only experiments with $\tilde{P}_c \geqslant 0{\cdot}5$ are described, since the relation between the estimators is the same for any $\tilde{P}'_c < 0{\cdot}5$ as for the one with $\tilde{P}_c = 1 - \tilde{P}'_c$.

The optimum arrangement for a bio-assay experiment is with dosages x symmetrically disposed around the value for which $\tilde{P} = 0{\cdot}5$, for it is with this arrangement that the variances of the estimates are smallest. Of course, without knowing the values of the parameters exactly an experiment cannot be accomplished with just this disposition of dosages, but by preliminary experimentation the position of x for which $\tilde{P} = 0{\cdot}5$ can be located approximately, and a well-designed experiment should not be done with central x very different from this value. In a sense, therefore, the comparison of the estimates with each other when $\tilde{P}_c = 0{\cdot}5$ is the most important one, since it compares the estimates in the ideal experiment from which good experiments will not be far removed. With this experimental arrangement, the estimate of α is unbiased, both for the minimum normit χ^2 and for the maximum likelihood estimator, whether it is estimated alone or simultaneously with β. For β, however, even in this arrangement, both estimators are biased, and for all other arrangements the estimate of each parameter is biased for both estimators. Unbiased estimation, therefore, is seen here to be exceptional, limited to α, and even in that case only with the model experiment.*

The results are shown in detail in Table 4 and Table 5. For the case of estimating α with β known (Table 4), the bias is positive for the maximum likelihood estimate, and negative for the minimum normit χ^2 estimate, the absolute value of the latter being the larger of the two. The variance is in all cases smaller for the minimum normit χ^2 estimate, and sufficiently smaller to outbalance the larger bias, the net result being a smaller mean square error for the minimum normit χ^2 estimate in all the experiments. In the case of α and β to be estimated simultaneously (Table 5), the bias is negative for α and positive for β, except for the minimum

* I do not mean to imply that unbiasedness is necessarily a desideratum. It is not obvious that, say, an unbiased estimator T_1 whose variance is larger than the mean square error of a biased estimator T_2, is preferable to it. I should prefer T_2.

normit χ^2 estimate, which at $\bar{P}_c = 0{\cdot}8$ is positively biased for α and negatively for β. The variance is smaller for the minimum normit χ^2 estimate, of either parameter, alone or both simultaneously, and with all dosage arrangements, and so also is the mean square error. It may be noted that whereas in the case of α alone to be estimated, the minimum normit χ^2 estimate achieves a smaller mean square error in spite of its bias having a larger absolute value than the maximum likelihood estimate, in the case of both α and β to be estimated, for

Table 4. *Estimate of α with β known*

Based on the total sampling population. The samples with infinite estimate by maximum likelihood are omitted in calculating all statistics. The greatest frequency of such samples is 0·04 %, occurring in the experiment with $\bar{P}_c = 0{\cdot}8$.

True P at dose			Bias		Variance		Mean square error		$1/I$	Mean square error $\div 1/I$	
Low	Mid.	High	Max. likelihood	Min. normit χ^2	Max. likelihood	Min. normit χ^2	Max. likelihood	Min. normit χ^2		Max. likelihood	Min. normit χ^2
0·3	0·5	0·7	0	0	0·0589	0·0537	0·0589	0·0537	0·0559	1·054	0·961
0·393	0·6	0·782	0·0068	−0·0073	0·0606	0·0546	0·0606	0·0547	0·0571	1·061	0·958
0·5	0·7	0·853	0·0152	−0·0177	0·0663	0·0572	0·0665	0·0575	0·0612	1·087	0·940
0·624	0·8	0·914	0·0287	−0·0401	0·0802	0·0598	0·0810	0·0614	0·0706	1·147	0·870

Table 5. *Estimate of α and β simultaneously*

Based on the total sampling population. The samples with infinite estimate by maximum likelihood are omitted in calculating all statistics, the frequency of such samples being shown in the last column.

True P at dose			Bias		Variance		Mean square error		% samples insoluble by max. likelihood
Low	Mid.	High	Max. likelihood	Min. normit χ^2	Max. likelihood	Min. normit χ^2	Max. likelihood	Min. normit χ^2	
					Estimate of α				
0·3	0·5	0·7	0	0	0·0675	0·0588	0·0675	0·0588	0·09
0·393	0·6	0·782	−0·0027	−0·0079	0·0880	0·0791	0·0880	0·0792	0·11
0·5	0·7	0·853	−0·0135	−0·0041	0·1587	0·1485	0·1589	0·1485	0·60
0·624	0·8	0·914	−0·0278	0·0517	0·3698	0·2953	0·3706	0·2980	4·43
					Estimate of β				
0·3	0·5	0·7	0·0467	0·0288	0·1109	0·0947	0·1131	0·0955	0·09
0·393	0·6	0·782	0·0505	0·0234	0·1172	0·0936	0·1198	0·0941	0·11
0·5	0·7	0·853	0·0588	0·0018	0·1339	0·0898	0·1374	0·0898	0·60
0·624	0·8	0·914	0·0526	−0·0564	0·1494	0·0824	0·1522	0·0856	4·43

a number of the dosage arrangements the absolute value of the bias as well as the variance is smaller for the minimum normit χ^2 estimate.

3. *Lower bound of mean square error*

In the case of one parameter to be estimated, with certain regularity conditions obtaining which are well fulfilled in the present problem, the lower bound for the mean square error of an estimate T of a parameter θ is given by*

$$E(T-\theta)^2 \geqslant \frac{(1+\partial b/\partial\theta)^2}{I} + b^2, \tag{17}$$

and the lower bound of the variance is the first term of the right-hand side of (17), where $b = E(T-\theta)$ is the bias of the estimate T, and I is Fisher's amount of extractable information, the value of which is

$$I = E\left(\frac{\partial \ln \phi}{\partial \theta}\right)^2, \tag{18}$$

where ϕ is the probability of the sample. The parameter to be estimated being α, the value of I is

$$I(\alpha) = \Sigma \frac{n_i Z_i^2}{P_i Q_i}. \tag{19}$$

For the evaluation of the lower bound (17), $I(\alpha)$ can be calculated directly, but $\partial b/\partial\alpha$ is also required. This was evaluated by plotting the bias b against α and estimating the slope of the bias function graphically, at required values of α.† The bias function for both estimators is shown in Fig. 2.

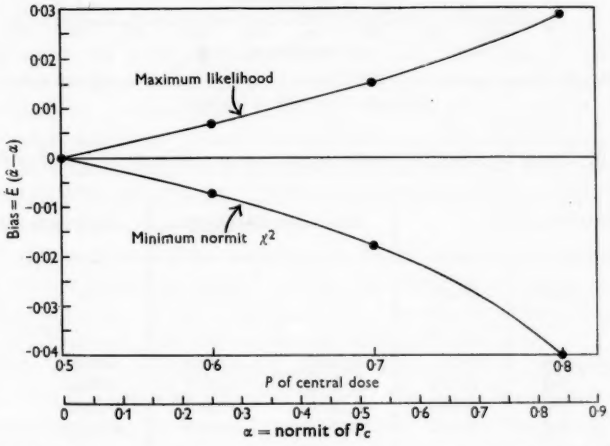

Fig. 2. Bias of $\hat{\alpha}$ in relation to α; estimate of α with β known. The evaluation of $\partial b/\partial\alpha$ required for calculation of the lower bound (17), was obtained graphically from an enlarged replica of this figure.

* This inequality is frequently referred to as the 'Cramér-Rao lower bound'. It was, however, developed by Fréchet, and in a very clear way, earlier than by these authors (Savage, 1954).

† Change of the dosage arrangement from one with $P_c = 0{\cdot}5$ to some other value of P_c is equivalent to retaining the original dosage disposition and changing α from $\alpha = 0$ to $\alpha =$ normit of P_c.

The lower bounds of the maximum likelihood estimator and the minimum normit χ^2 estimator are given in Table 6, and shown in Fig. 3 together with the corresponding mean square errors. It is seen that neither estimate attains the lower bound for its mean square error, but that the lower bound of the minimum normit χ^2 estimator is lower than that of the maximum likelihood estimator. The attainment of a smaller lower bound by the minimum normit χ^2 estimator is directly related to the character of its bias function, the first derivative of which is everywhere negative, while that of the maximum likelihood estimator is everywhere positive (Fig. 2).

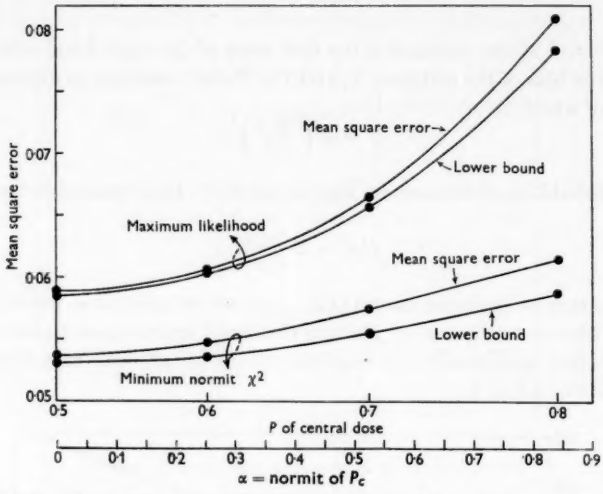

Fig. 3. Mean square error and lower bound of the mean square error; estimate of α with β known.

Table 6. *Lower bound of the mean square error; estimate of α with β known*

True P at dose			Maximum likelihood		Minimum normit χ^2	
Low	Mid.	High	L.B.	M.S.E.	L.B.	M.S.E.
0·3	0·5	0·7	0·0585	0·0589	0·0531	0·0537
0·393	0·6	0·782	0·0603	0·0606	0·0536	0·0547
0·5	0·7	0·853	0·0656	0·0665	0·0555	0·0575
0·624	0·8	0·914	0·0784	0·0810	0·0587	0·0614

We may also note the relation between the variance and $1/I$, which is sometimes treated as the variance equivalent of the amount of extractable information and is in fact the lower bound of the variance for an unbiased estimator. At all the dosage arrangements of the experiments, the variance of the minimum normit χ^2 estimate is smaller and that of the maximum likelihood estimator is larger than $1/I$ (Table 4).

4. *Distribution of* χ^2

Related to estimation of the parameters in the situation of bio-assay is the testing of goodness of fit by the chi-square test. The standard practice is to calculate the classic χ^2 of Karl Pearson

$$\chi^2 = \Sigma \frac{(o_i - e_i)^2}{e_i}, \tag{20}$$

where, in the present situation, o_i is the observed number of responses at x_i, and e_i is the 'expected' number, computed as $e_i = \hat{P}_i n_i$, where $\hat{P}_i$ is the probability at x_i calculated from (1) using the estimated values of the parameters to replace the true values. The test of significance is obtained by reference to a table of the distribution of chi-square* for degrees of freedom equal to the number of dosages x_i less the number of parameters which have been estimated.

It is well known that the χ^2 of (20) is only asymptotically distributed as chi-square, and then only if the estimates of the parameters are asymptotically efficient estimates. For finite samples it is so distributed only approximately. Now the classic χ^2 of Pearson is not the only function of the observations which is asymptotically distributed as chi-square. The normit χ^2 of (4),

$$\chi^2(\text{normit}) = \Sigma n_i w_i (X_i - \hat{X}_i)^2,$$

which is the quantity minimized to obtain the minimum normit χ^2 estimate, is also asymptotically distributed as chi-square. If the parameters have been estimated by minimum normit χ^2, the normit χ^2 is somewhat easier to compute than the Pearson χ^2, since it can be calculated directly using the estimated parameters by the familiar formula

$$\chi^2 = \Sigma n_i w_i (X_i - \bar{X})^2 - \hat{\beta} \Sigma n_i w_i (X_i - \bar{X})(x_i - \bar{x}). \tag{21}$$

When the parameters have been estimated in this way, therefore, it seems natural to calculate the normit χ^2 for the chi-square test, and the question arises as to whether it is distributed as closely to the asymptotic chi-square distribution as is the Pearson χ^2. Then again, since the minimum normit χ^2 estimate and the maximum likelihood estimate are both asymptotically efficient, the question arises as to whether perhaps one rather than the other, used to calculate the expected numbers e_i in the Pearson χ^2 of (20), gives a better approximation of the theoretical chi-square distribution.

In order to shed some light on these questions, the distributions of three sample χ^2 functions were calculated for the experiment with $\tilde{P}_c = 0{\cdot}5$ in the case with α alone to be estimated: the normit χ^2 for the minimum normit χ^2 estimates, the Pearson χ^2 for the minimum normit χ^2 estimates, and the Pearson χ^2 for the maximum likelihood estimates.

For each sample, the χ^2 was computed, using the estimates with four decimal places and writing the calculated χ^2 to three decimal places. The result was punched into a card together with the sample designation and its probability. The distribution of the χ^2 was then easily obtained by cumulating the probabilities progressively on the tabulator, with the samples ordered on the value of the χ^2. The cumulative distributions for the sample χ^2 functions are shown in Fig. 4 together with the asymptotic chi-square distribution, and some selected values are listed numerically in Table 7.

* By the distribution of chi-square we mean the exact distribution of the sum of squares of a number of independent normal deviates having unit standard deviations.

Table 7. *Distribution of sample χ^2 functions for experiment with $P_c = 0{\cdot}5$, α alone to be estimated; comparison of the percentage frequencies with the theoretical frequencies*

The entries give in % for the respective χ^2's, $\Pr(\chi^2 \geqslant \chi_0^2)$, where in the theoretical asymptotic chi-square distribution for 2 degrees of freedom, $\Pr(\chi^2 \geqslant \chi_0^2) = L\,\%$.

Level L (%)	χ_0^2	Max. likelihood estimate	Min. normit χ^2 estimate	
		Pearson χ^2	Pearson χ^2	Normit χ^2
25	2·773	26·75	26·66	24·02
20	3·219	22·69	22·22	20·96
15	3·794	15·65	15·49	11·76
10	4·605	9·65	9·64	8·15
5	5·992	5·14	5·12	2·70
1	9·210	0·79	0·81	0·39
0·5	10·597	0·46	0·46	0·14

Inspection of the normit χ^2 distribution in Fig. 4 and Table 7 reveals rather wide discrepancy of the distribution frequencies from those of the asymptotic chi-square distribution. The frequencies for specific points of χ_0^2 are sometimes higher, sometimes lower than the asymptotic frequency, until about the 15 % level. From that point forward the normit χ^2 frequency is consistently too low; at the 5% level the frequency is 2·7%, and at the 1% level the frequency is 0·4 %. If, therefore, the normit χ^2 were used for a test at either of these levels, the tested hypothesis would be 'rejected' considerably less frequently than required by theory. The Pearson χ^2, for the same minimum normit χ^2 estimates, is seen to be in much better agreement with the asymptotic chi-square distribution in the region below the 15% level. For the 5 % level the frequency is 5·1 % and for the 1 % level it is 0·8 %. The normit χ^2 should therefore not be used in preference to the Pearson χ^2, for a χ^2 test of significance, even though it is somewhat easier to calculate. This conclusion is in line with that of David (1950), who condemned the use of the Neyman reduced χ^2 for a chi-square test of significance, even when there was good reason for preferring an estimate obtained by minimizing that χ^2.

Comparison of the distribution of the Pearson χ^2 in Fig. 4 and Table 7, for the minimum normit χ^2 estimate and the maximum likelihood estimate, shows them to be closely similar in their approximation of the asymptotic chi-square distribution. At the 5 % and the 1 % levels the approximation is somewhat closer for the minimum normit χ^2 estimate than for the maximum likelihood estimate. Thus, the advantage of the minimum normit χ^2 estimate over the maximum likelihood estimate found in respect of the variance and mean square error is supported, or at least not offset, by the approximation of its χ^2 distribution to the asymptotic distribution.

5. *Conclusion*

It has been mathematically demonstrated (Taylor, 1953*a*) that the minimum normit χ^2 estimate falls in the class R.B.A.N. (regular best asymptotically normal) of Neyman. The estimate is therefore asymptotically efficient, and in 'large sample theory' it is equivalent to the maximum likelihood estimate. In large samples, as this phrase commonly is used in

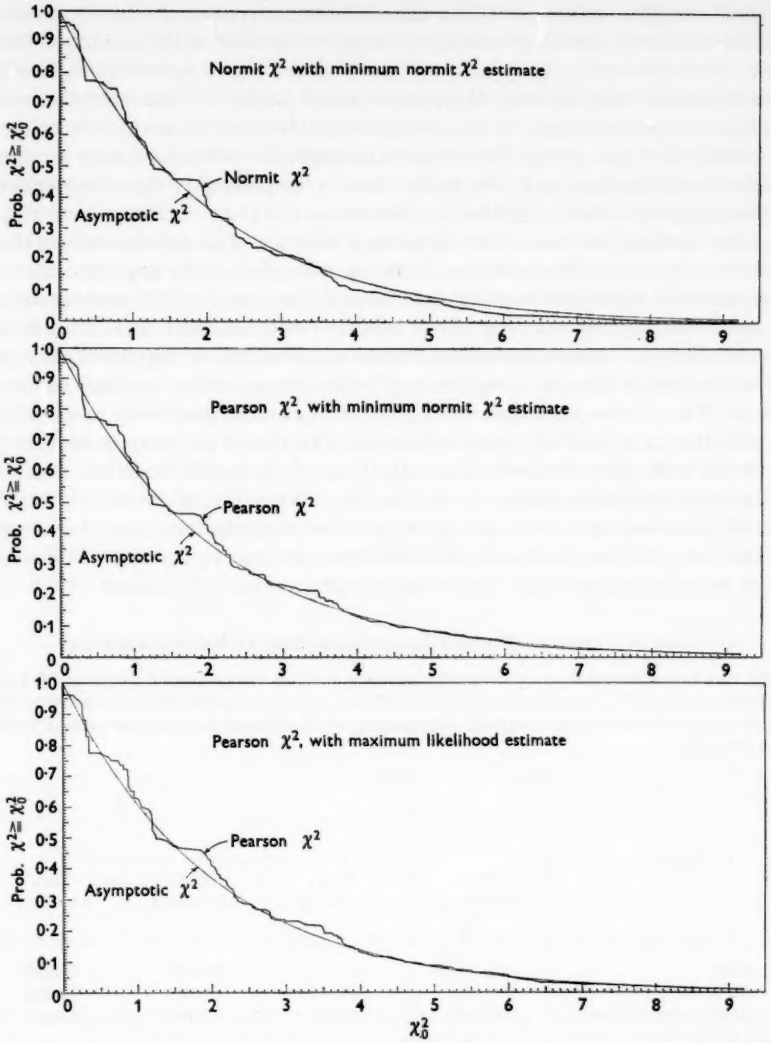

Fig. 4. Distribution of sample χ^2 functions; estimate of α with β known; the theoretical asymptotic distribution is the chi-square distribution for 2 degrees of freedom.

statistical literature, both estimates are normally distributed, with mean equal to the parameter estimated and variance given by

$$\sigma^2(\hat{\alpha}) = 1/\Sigma(n_i Z_i^2/(P_i Q_i)), \tag{22}$$

$$\sigma^2(\hat{\beta}) = 1/\Sigma(n_i Z_i^2/(P_i Q_i))\,(x_i - \bar{x})^2, \tag{23}$$

where

$$\bar{x} = \Sigma(n_i Z_i^2/(P_i Q_i))\,x_i/\Sigma(n_i Z_i^2/(P_i Q_i)). \tag{24}$$

For small samples, we have found in the experiments presented that the mean square error of the minimum normit χ^2 estimate is smaller than that of the maximum likelihood estimate. Since this conclusion is based on numerical findings in specific situations and not on a mathematical demonstration, the question may be asked whether it can be considered proved, and how general it may be regarded as being. Of course no specific numerical results can be considered to be a proof. Even so far as concerns the particular dosage arrangements for which the calculations have been made, there is the possibility that some arithmetical errors have been committed. All that can be said on this point is that it is unlikely in the extreme that arithmetical errors have been made that are of so serious a nature that their correction would reverse this conclusion. Care has been taken, so far as practicable, to check each arithmetical operation involved in the calculation, and in addition over-all indirect checks have been applied whenever the possibility presented itself. The ratios of and differences between the mean square errors of the two estimates, as they have emerged from these calculations, change in a regular way with change in the assumed values of the parameters. The conclusions are essentially the same as those previously reached independently with the use of carefully planned samples. The results are in agreement, in general and in detail, with those obtained in investigations of the logistic function, which is very similar in curvature characteristics to the integrated normal function. So far, therefore, as the specific experimental conditions in which these investigations have been made are concerned, there seems no reasonable doubt but that the mean square error of the maximum likelihood estimate is larger than that of the minimum normit χ^2 estimate.

Table 8. *Estimate of α and β simultaneously, with dosages spread*

$P_c = 0{\cdot}5$, the lower dosage is at $P_i = 0{\cdot}1$, the upper is at $P_i = 0{\cdot}9$ instead of at 0·3 and 0·7 as in the standard experiments. The samples with infinite estimate by maximum likelihood constitute 12·2 % of the sampling population and are omitted in calculating all statistics. The results are based on the total sampling population.

Statistic	$\hat{\alpha}$		$\hat{\beta}$	
	Max. likelihood	Min. normit χ^2	Max. likelihood	Min. normit χ^2
Bias	0	0	0·0507	−0·0983
Variance	0·1052	0·0562	0·1246	0·0566
Mean square error	0·1052	0·0562	0·1272	0·0663

A further question that may be asked is whether the same conclusion applies to experiments with a different arrangement of dosages. It was early suggested in connexion with the logistic function that the relation between the mean square errors of the two estimates might be reversed, if the dosages were arranged with a wider 'spread'. For this reason a calculation was made, using the total population of samples, for an experiment with $P_c = 0{\cdot}5$, but with the lower dosage at $P_i = 0{\cdot}1$ and the upper dosage at $P_i = 0{\cdot}9$, instead of at 0·3 and 0·7, respectively. The results (Table 8) not only confirmed the larger mean square error of the maximum likelihood estimate, but showed it to be in larger ratio to that of the

minimum normit χ^2 than with the more narrowly disposed dosages, and moreover pointed out more emphatically the difficulty encountered in the maximum likelihood estimator's yielding infinite estimates.

Similarly, it was suggested that a different sort of estimate, such as a bio-assay estimate of relative potency, might show the smaller error to be with the maximum likelihood estimator. A calculation was therefore made for an experiment simulating a four-point parallel assay. In such an experiment a mixture of unknown strength is to be assayed as to its relative potency with respect to another mixture considered as standard. For the standard mixture, two concentrations x_1 and x_2 measured in logarithmic units are used, and n_i animals exposed with each; for the unknown, the same two concentrations x_1 and x_2 are used, and n_i exposed at each. Thus there are four relative frequencies observed, two corresponding to the standard mixture, two to the unknown. Two normit lines $(\alpha + \beta x_i)$ are fitted, one to the points of the standard mixture, one to the points of the unknown, these lines constrained to be parallel, so that $\hat{\beta}$ is the same for both. The distance between the fitted parallel lines, in units of x, is the difference of the logarithms of the dosage of the unknown and standard respectively, which produces the same response. This is taken as the measure of the logarithm of the relative potency, and given by $M = (\hat{\alpha}_u - \hat{\alpha}_s)/\hat{\beta}$, where $\hat{\alpha}_u$ is the estimate of α corresponding to the normit line of the unknown, and $\hat{\alpha}_s$ is the similar estimate corresponding to the line of the standard mixture (see Finney, 1947). The estimate, based on 100 stratified random samples, showed the maximum likelihood estimator to have the larger mean square error (Table 9).

Table 9. *Estimate of log relative potency M, with a four-point parallel assay*

Simulated experiment with 'standard' and 'unknown' taken as of equal potency, $M = 0$. For both, lower dosage x_1 at $P_i = 0{\cdot}3$, upper dosage x_2 at $P_i = 0{\cdot}7$. $n_i = 10$ for each of the four observations. Based on 100 stratified random samples of the four-point assay.

Estimator	Bias*	Variance	M.S.E.
Maximum likelihood	−0·014	0·185	0·185
Minimum normit χ^2	−0·013	0·175	0·176

* The expected value is actually zero; the empirical figures are to be attributed to sampling fluctuation.

Lastly, it was suggested that the relative results would be reversed if the number of doses were greatly increased. In this connexion specific results can be quoted only for experiments with the logistic function. Since in other experiments performed with both functions the results are similar, these findings with the logistic function are relevant. Experiments comparing the maximum likelihood estimator and the minimum logit χ^2 estimator, with dosages up to 11 (Berkson, 1955*b*), and experiments comparing the maximum likelihood estimator with the minimum Pearson χ^2 estimator with dosages up to 100 (Taylor, 1953*b*), showed the maximum likelihood estimator to have the larger mean square error.

In spite of all the corroboratory findings, in a variety of experimental conditions, of the larger mean square error of the maximum likelihood estimator, the possibility is not excluded that some conditions may be definable in which the minimum normit χ^2 estimator has the

larger error. It seems unlikely, however, that if such conditions are discovered, they will correspond to any met in practice in a well-designed bio-assay or in similar assay.

Even with all this, the important finding of the present investigation is not that the minimum normit χ^2 estimator has a smaller error than the maximum likelihood estimator, even if this is accepted in full generality, but rather that the maximum likelihood estimator does *not* have the smaller error in all circumstances. This last really did not require proof, since no serious evidence to the contrary had ever been presented. What does require wider appreciation among statistical writers is that it is unwarranted to claim that some presently known principle of estimation necessarily always yields the best estimate. If in the present circumstances the minimum normit χ^2 estimate is easier to compute, and at the same time is a better estimate than that provided by the maximum likelihood estimator, there may be other methods of estimate—and I believe there are— which are even easier to compute and, in the practical circumstances in which they are applied, may be better than the minimum normit χ^2 estimator.

To the many young ladies and gentlemen, who, over the last 8 years, have helped in the calculations required for the investigations referred to in this paper, I tender my thanks. Some, but not all, of the individuals are: Gretchen Eusterman, Gudula Fischer, Marjorie Heiges, Bertram Haines, Ann Hogben, David Hogben, Richard Kersten, Beverley Newcomb, Fred Schwarz, David Sperling. Especially I owe gratitude to Mrs Helen Golenzer, who has been with the work from the beginning, and whose talent for orderly filing, and clairvoyant ability to locate any desired sheet, no matter how falsely identified by me, made progress possible, and prevented Bedlam from supplanting the Mayo Clinic.

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SHORTER INTERVALS FOR THE PARAMETER OF THE BINOMIAL AND POISSON DISTRIBUTIONS

By W. L. STEVENS*

SUMMARY. Different methods are examined for determining, for the binomial and Poisson parameters, narrower intervals than those furnished by the classical method. A way is found for calculating, from existing tables, the limits which are produced by the device of adding, to the observation, a randomly chosen number between 0 and 1.

1. Introduction

Some recent papers (Sterne, 1954; Crow, 1956) show that statisticians are concerning themselves with the problem of how to construct intervals for the parameter of the binomial distribution, narrower than those found by the usual method (Clopper & Pearson, 1934; Fisher & Yates, 1943, Table VIII, I). An interesting solution to this problem, with the necessary tables, has appeared in the paper by Crow. It is, however, to be questioned whether the proposed method (which is based on re-ordering the significance of observations in the tails) is satisfactory for all purposes. The intervals are indeed narrower than those found by the usual method, and it is true that the probability that the interval will cover the true value is not less than that declared. We believe, however, that one often requires more than this: one wants to know separately the probabilities or bounds to the probabilities, that the value of the parameter is below the lower limit or above the upper limit. This is certainly true when we are interested in only one limit—a case which is quite common in practice; for example, a purchaser of electric lamps is interested only in the upper limit of the percentage defective. It would appear that no statements of the separate probabilities that the value of the parameter lies below the lower limit or above the upper can be made in respect of the intervals tabulated by Crow.

There exists, however, another and perhaps more elegant method for constructing shorter intervals (Anscombe, 1948; Stevens, 1950; Tocher, 1950; Eudey, 1949; Pearson, 1950), which possess the two desirable properties: (1) that statements of probabilities are equalities instead of inequalities, and (2) that the probabilities are known, separately of the value of the parameter lying below the lower limit or above the upper limit. Even more, Blank (1956) has shown that the method possesses theoretically optimal properties.

The method, stated briefly in relation to the binomial, though applicable to any discontinuous distribution, is as follows: we draw n balls from an infinite population of black and white balls and find that f of them are black. We then draw a number x from the rectangular distribution, zero to one. Then $y = f + x$ is a real number between 0 and $n + 1$, from which we can find π_0 and π_1, respectively, the lower and upper limits to the parameter π, the proportion of black balls in the population. The probabilities that the value of π lies below the lower limit, between the two, or above the upper limit, will be known exactly.

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2. CALCULATION OF LIMITS

The functional relation between π_0, the lower 10 % limit, and y, for $n = 10$, is illustrated in the graph, based on a table in Stevens (1950). It is seen to be composed of arcs concave upwards linking the points corresponding to integral values of y. The function is continuous but the first derivative discontinuous at these points.

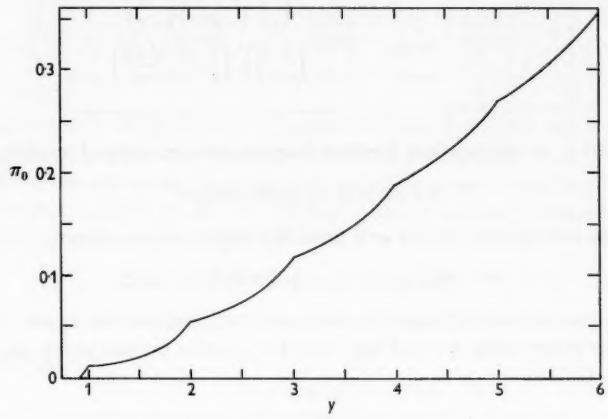

Fig. 1. Portion of the graph showing relation between lower 10 % limit ($n = 10$) and $y = f + x$.

We shall continue the discussion on terms of the lower limit: the necessary change of words to apply it to the upper limit will be obvious. Now a simple first suggestion for finding a limit, without using a table, would be to find, by the usual method, limits π_0 and π_0', corresponding respectively to the observations, f in n and $f+1$ in n, and to make a random linear interpolation between these two values. This is equivalent to replacing the arcs in the graph by straight lines joining their extremities. But it is immediately seen that the consequence of doing this is that the probability that π lies below the lower limit will be *not less* than that stated; though admittedly it will not be much more. Now, if we have to make an inequality statement, this must be in the opposite form, i.e. that the probability is *not greater* than that stated. Hence, this method cannot be considered satisfactory.

We therefore turn to consider the possibility of interpolating between two consecutive integral values of y ($y = f$ and $y = f+1$, i.e. $x = 0$ and $x = 1$) by means of a Taylor's series. The value of $d\pi/dx$ (we will drop the suffix, 0) is obtained from Stevens, equation (2·22); we should note, however, a mistake in equation (2·13)—the h_0 and h_1 should be interchanged. We find

$$\frac{d\pi}{dx} = \frac{1}{\frac{(n-f)x}{1-\pi} + \frac{f(1-x)}{\pi}}.$$

The second derivative is found in the usual way from

$$\frac{d^2\pi}{dx^2} = \frac{\partial}{\partial x}\left(\frac{d\pi}{dx}\right) + \frac{\partial}{\partial \pi}\left(\frac{d\pi}{dx}\right)\left(\frac{d\pi}{dx}\right),$$

the calculation being simplified by the fact that it is required only at the values $x = 0$ and $x = 1$. The results are

x	$d\pi/dx$	$d^2\pi/dx^2$
0	$\frac{\pi}{f}$	$\left(\frac{\pi}{f}\right)^2\left(\frac{f+1}{\pi}-\frac{n-f}{1-\pi}\right)$
1	$\frac{1-\pi}{n-f}$	$\left(\frac{1-\pi}{n-f}\right)^2\left(\frac{f}{\pi}-\frac{n-f+1}{1-\pi}\right)$

If x is less than $\frac{1}{2}$, we extrapolate forward from the lower value of π, using the formula

$$\pi + (d\pi/dx)\,x + \tfrac{1}{2}(d^2\pi/dx^2)\,x^2$$

and if greater, we extrapolate backwards from the higher value, using

$$\pi - (d\pi/dx)\,(1-x) + \tfrac{1}{2}(d^2\pi/dx^2)\,(1-x)^2.$$

To illustrate the process, we will consider the lower 10 % limit for the result '3 in 10'. From Fisher & Yates or other table, we find the lower 10 % limits for this result and for the result '4 in 10'. We find

x	π	$d\pi/dx$	$\frac{1}{2}(d^2\pi/dx^2)$
0	0·116	0·0387	0·0199
1	0·188	0·116	0·0411

The results of interpolation can now be compared with the tabulated values as follows:

x	(*a*)	Table	(*b*)
0·0	0·116	0·116	—
0·1	0·120	0·120	—
0·2	0·125	0·124	—
0·3	0·129	0·129	0·127
0·4	0·135	0·135	0·133
0·5	0·140	0·141	0·140
0·6	0·146	0·148	0·148
0·7	0·153	0·157	0·157
0·8	—	0·166	0·166
0·9	—	0·176	0·177
1·0	—	0·188	0·188

(*a*) Extrapolated forward from $x = 0$. (*b*) Extrapolated backward from $x = 1$

The errors of interpolation would seem to be too small to be revealed by this comparison.

To find the upper 10% limit, we find from the table the limits for the observed result '3 in 10' and for the result '2 in 10'. These correspond respectively to $x = 1$ and $x = 0$. Hence

x	π	$d\pi/dx$	$\frac{1}{2}(d^2\pi/dx^2)$
0	0·450	0·150	−0·0432
1	0·552	0·064	−0·0255

For $x = 0{\cdot}5$ both forward and backward extrapolation yield $\pi = 0{\cdot}514$, while the table gives $\pi = 0{\cdot}512$. The difference probably represents the error in the table.

It is advisable to use the same value of x for both limits and indeed for all intervals, if more than one is determined. Thus suppose the table of random numbers yields 268..., which on rounding gives $x = 0{\cdot}27$. The limits will be

$$\pi_0 = 0{\cdot}116 + (0{\cdot}039)(0{\cdot}27) + (0{\cdot}020)(0{\cdot}27)^2 \quad = 0{\cdot}128,$$

$$\pi_1 = 0{\cdot}450 + (0{\cdot}150)(0{\cdot}27) + (-0{\cdot}043)(0{\cdot}27)^2 = 0{\cdot}487$$

and we can state that

$$\Pr(\pi < 0{\cdot}128) = 10\%$$

$$\Pr(0{\cdot}128 < \pi < 0{\cdot}487) = 80\%$$

$$\Pr(0{\cdot}487 < \pi) = 10\%.$$

Special treatment is needed for $f = 0$ and $f = n$. It is recalled that if $f = 0$ and $x < 1 - P$, then $\pi = 0$. Hence, if such a value of x is drawn (e.g. $x < 0{\cdot}9$, for a lower 10% limit) π_0 is taken as zero. Otherwise, we extrapolate backwards from the limit corresponding to $f = 1$.

Finally, we may note that essentially the same method may be used for setting limits to μ, the parameter of the Poisson distribution. The derivatives at $x = 0$ and 1 are:

x	$d\mu/dx$	$d^2\mu/dx^2$
0	$\frac{\mu}{f}$	$\left(\frac{\mu}{f}\right)^2\left(\frac{f+1}{\mu}-1\right)$
1	1	$\left(\frac{f}{\mu}-1\right)$

3. CONCLUSION

It is open to question whether anything material is gained by devices for finding intervals shorter than those provided by the classical method. If, however, the statistician is determined to have them, it would seem that, for both practical and theoretical reasons, the device of adding to the frequency a random number between zero and one is superior to any other. This paper has shown that these intervals can be found by a simple interpolation in existing tables.

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UPPER PERCENTAGE POINTS OF THE GENERALIZED BETA DISTRIBUTION. II

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1. Introduction

These tables extend the tabulation of the 80, 85, 90, 95 and 99% points of $I_x(k; p, q)$ to the case $k = 3$. They are a continuation of the tables for $k = 2$ given in Foster & Rees (1957), which will be referred to as 'I'. Reference is made to I for all definitions.

2. Interpolation

Interpolation will be required for integer values of $\nu_1 = 2q + 2$. For ν_1 greater than about 80, linear interpolation will give accuracy to one unit in the fourth decimal place; for smaller values of ν_1, 4-point interpolation to halves should be used. For interpolation between $\nu_1 = 194$ ($q = 96$) and $\nu_1 = \infty$, 3-point harmonic interpolation, based on $q = 48$, $q = 96$ and $q = \infty$, may be used, as indicated in I.

3. Uses of the tables

Typical applications to tests of significance for the case $k = 2$ were given in I. A further example is given below to illustrate the present tables.

Example. Table 1 gives the analysis of dispersion for the three characters, head length (x_1), height (x_2) and weight (x_3), measured on 140 schoolboys of almost the same age, belonging to six different schools in an Indian city. This example is taken from Rao (1952, p. 263) who examines the data using the alternative Λ criterion of Pearson & Wilks.

Table 1

	D.F.	Sums of products matrix					
			x_1^2	x_2^2	x_3^2	x_1x_2	x_1x_3	x_2x_3
Between schools	5	Q_{ij}	752·0	151·3	1,612·7	214·2	521·3	401·2
Within schools	134	W_{ij}	12,809·3	1,499·6	21,009·6	1,003·7	2,671·2	4,123·6
Total	139	S_{ij}	13,561·3	1,650·9	22,622·3	1,217·9	3,192·5	4,524·8

It is required to test whether there are any significant differences in boys' physique between schools. On the null hypothesis of no differences, the matrices (Q_{ij}) and (W_{ij}), divided by their degrees of freedom, are independent estimates of the same parent dispersion matrix, and our test consists in computing the greatest latent root of $|Q - \theta S| = 0$,

and entering the table with $\nu_1 = 134$ and $\nu_2 = 5$. We find that $\theta_{\max.} = 0{\cdot}10055$, which is only significant at the 15 % level.

4. Method of computation

The computations were carried out on the DEUCE Computer of the English Electric Company. Let $\theta_{\max.}$ denote the greatest root of

$$|\nu_2 B - (\nu_1 A + \nu_2 B)| = 0,$$

where A and B are independent estimates, based on ν_1 and ν_2 degrees of freedom, of a parent dispersion matrix of a trivariate Normal population. Define

$$I_x(3; p, q) = \Pr\{\theta_{\max.} \leqslant x\},$$

where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(|\nu_1 - 3| + 1)$. Roy (1945) gives a formula which, in the present notation, may be written

$$I_x(3; p, q) = K'\{2B_x(p, q)\, B_x(2p+2, 2q) - 2B_x(p+1, q)\, B_x(2p+1, 2q) - x^{p+1}(1-x)^q B_x(2; p, q)\},$$

where $B_x(p, q)$ is the Incomplete Beta function;

$$K' = \frac{1}{2}\frac{p+q}{p+q+1}\frac{1}{B(p, q)}\frac{\Gamma(2p+2q+3)}{\Gamma(2p+1)\,\Gamma(2q+1)};$$

and $B_x(2; p, q) = I_x(2; p, q)/K$, as defined in I. As it stands, this formula is too ill-conditioned to be useful for computation. If, however, we substitute for $B_x(2; p, q)$ and distribute the normalizing constant, K', we obtain

$$\begin{aligned} I_x(3; p, q) = {} & I_x(2p+2, 2q)\, I_x(p, q) + p/q(2p+2q+1)\{I_x(2p+2, 2q)\, I_x(p, q) \\ & - I_x(2p+1, 2q)\, I_x(p+1, q) + b_x(2p, 2q)\, I_x(p, q) - b_x(p, q)\, I_x(2p, 2q)\}, \end{aligned}$$

where $I_x(p, q)$ is the Beta distribution, and

$$b_x(p, q) = x^{p+1}(1-x)^q \frac{\Gamma(p+q+1)}{p\Gamma(p+1)\,\Gamma(q)}.$$

The ranges of p and q were $p = \frac{1}{2}(\frac{1}{2})4$, $q = 1(1)96$. For the integral values of p, $I_x(p, q)$ was computed recursively as in I, and $b_x(p, q)$ by means of the relations

$$\begin{aligned} b_x(p, 0) &= 0 && (p > 0), \\ b_x(1, 1) &= 2x^2(1-x), \\ b_x(1, q+1) &= b_x(1, q)\,(q+2)/q && (q > 0), \\ b_x(p, q) &= \frac{(p+q)\,(p-1)}{(p+q)\,(p-1)+1}\{(1-x)\, b_x(p, q-1) + x b_x(p-1, q)\} && (p, q > 0). \end{aligned}$$

For the half-integral values of p, $I_x(p, q)$ was computed recursively by means of

$$\begin{aligned} I_x(\tfrac{1}{2}, 1) &= x^{\frac{1}{2}}, \\ I_x(p+1, 1) &= x I_x(p, 1) && (p > 0), \\ I_x(\tfrac{1}{2}, q+1) &= \frac{1}{2q}\{I_x(\tfrac{1}{2}, q) - I_x(\tfrac{3}{2}, q)\} + I_x(\tfrac{1}{2}, q) && (q > 0), \\ I_x(p, q) &= (1-x)\, I_x(p, q-1) + x I_x(p-1, q) && (p, q > 0), \end{aligned}$$

and $b_x(p, q)$ was computed by means of

$$b_x(\tfrac{1}{2}, 1) = 3x^{\frac{1}{2}}(1-x),$$
$$b_x(\tfrac{1}{2}, q+1) = b_x(\tfrac{1}{2}, q)\,(1-x)\,(2q+3)/2q,$$
$$b_x(p, 0) = 0,$$

together with the last formula for $b_x(p, q)$ given above.

As projected in I, the percentage points were obtained directly without prior tabulation of the distribution function. The method used (which may be described in more detail elsewhere) was to compute a table of $I_x(3; p, q)$ for the whole range of p and q for a sequence of values of x, starting at 0 and finishing at 1 at a fixed interval, h. The tables for the current value of x and the three preceding values, $x-h$, $x-2h$, $x-3h$, were stored on the drum. On completion of each current table, a 4-point inverse interpolation was carried out for each of the five required percentage points on the four values in position (p, q) on the four tables for all positions (p, q) such that $I_{x-2h}(3; p, q)$ was less than, and $I_{x-h}(3; p, q)$ was greater than the required percentage value. The resulting value was punched out on a card together with the corresponding p and q and an indication of the percentage value. The values were in fact punched out twice on each card, once rounded to four decimal places and once to six decimal places. Differencing was carried out on the latter values as a check on accuracy. The cards were subsequently sorted and the rounded values tabulated in the required layout. An interval $h = 2^{-8}$ was found to give accuracy to at least four decimal places for most of the percentage points. Some of the 99% points near the upper limit, $x = 1$, had to be obtained using a smaller interval of 2^{-10}.

The same method of computation could be used to obtain values for $\nu_2 > 10$, should the need for these be felt. At this stage, however, it was thought more useful to devote the computational effort to obtaining a fuller tabulation for ν_1.

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Generalized Beta distribution: 100P % points for x

ν_1 \ ν_2		3	4	5	6	7	8	9	10
	P								
4	0·80	0·9516	0·9635	0·9707	0·9755	0·9790	0·9816	0·9836	0·9852
	·85	·9645	·9733	·9786	·9821	·9846	·9865	·9880	·9892
	·90	·9769	·9826	·9861	·9884	·9900	·9913	·9922	·9930
	·95	·9887	·9915	·9932	·9943	·9951	·9957	·9962	·9966
	·99	·9978	·9983	·9987	·9989	·9990	·9992	·9993	·9993
6	0·80	0·8294	0·8629	0·8852	0·9012	0·9132	0·9226	0·9302	0·9364
	·85	·8561	·8846	·9036	·9171	·9273	·9352	·9416	·9468
	·90	·8857	·9087	·9238	·9346	·9427	·9490	·9540	·9582
	·95	·9218	·9378	·9482	·9556	·9612	·9655	·9689	·9717
	·99	·9664	·9734	·9779	·9811	·9835	·9853	·9868	·9880
8	0·80	0·7180	0·7639	0·7965	0·8210	0·8400	0·8554	0·8680	0·8786
	·85	·7497	·7911	·8203	·8421	·8591	·8728	·8840	·8933
	·90	·7871	·8229	·8480	·8667	·8812	·8929	·9024	·9103
	·95	·8365	·8646	·8842	·8986	·9098	·9188	·9261	·9322
	·99	·9086	·9247	·9359	·9441	·9504	·9554	·9595	·9628
10	0·80	0·6281	0·6798	0·7181	0·7478	0·7717	0·7913	0·8077	0·8217
	·85	·6610	·7090	·7443	·7716	·7935	·8114	·8264	·8392
	·90	·7008	·7440	·7757	·8000	·8195	·8353	·8486	·8598
	·95	·7560	·7922	·8185	·8386	·8546	·8676	·8784	·8876
	·99	·8439	·8679	·8852	·8983	·9086	·9170	·9239	·9298
12	0·80	0·5564	0·6102	0·6513	0·6840	0·7108	0·7332	0·7523	0·7688
	·85	·5887	·6396	·6783	·7089	·7340	·7549	·7726	·7879
	·90	·6287	·6757	·7112	·7392	·7620	·7810	·7971	·8109
	·95	·6857	·7266	·7574	·7815	·8010	·8172	·8309	·8426
	·99	·7816	·8113	·8334	·8505	·8643	·8757	·8853	·8934
14	0·80	0·4986	0·5525	0·5946	0·6289	0·6574	0·6816	0·7025	0·7207
	·85	·5297	·5813	·6215	·6541	·6811	·7039	·7236	·7407
	·90	·5687	·6172	·6548	·6851	·7101	·7313	·7494	·7652
	·95	·6254	·6689	·7024	·7292	·7512	·7698	·7857	·7994
	·99	·7245	·7582	·7837	·8040	·8206	·8344	·8462	·8564
16	0·80	0·4512	0·5042	0·5464	0·5812	0·6106	0·6359	0·6579	0·6773
	·85	·4809	·5321	·5728	·6062	·6343	·6584	·6794	·6978
	·90	·5185	·5672	·6057	·6372	·6637	·6863	·7058	·7230
	·95	·5739	·6185	·6535	·6820	·7058	·7261	·7436	·7589
	·99	·6735	·7096	·7376	·7601	·7788	·7947	·8083	·8201
18	0·80	0·4118	0·4633	0·5050	0·5398	0·5696	0·5954	0·6181	0·6383
	·85	·4401	·4902	·5307	·5643	·5930	·6179	·6396	·6589
	·90	·4760	·5242	·5629	·5950	·6222	·6457	·6663	·6845
	·95	·5296	·5745	·6103	·6398	·6647	·6861	·7048	·7212
	·99	·6282	·6657	·6953	·7195	·7398	·7571	·7721	·7853
20	0·80	0·3787	0·4284	0·4692	0·5037	0·5334	0·5595	0·5825	0·6031
	·85	·4054	·4542	·4940	·5276	·5564	·5816	·6038	·6237
	·90	·4397	·4870	·5254	·5576	·5852	·6093	·6304	·6493
	·95	·4914	·5359	·5719	·6019	·6275	·6497	·6692	·6864
	·99	·5879	·6262	·6568	·6821	·7035	·7220	·7381	·7523
22	0·80	0·3504	0·3983	0·4380	0·4719	0·5014	0·5274	0·5505	0·5714
	·85	·3758	·4230	·4619	·4951	·5238	·5491	·5715	·5917
	·90	·4085	·4545	·4924	·5244	·5521	·5764	·5979	·6172
	·95	·4580	·5019	·5377	·5679	·5938	·6165	·6365	·6544
	·99	·5521	·5906	·6218	·6478	·6700	·6893	·7063	·7214

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1 \ ν_2	P	3	4	5	6	7	8	9	10
24	0·80	0·3259	0·3720	0·4106	0·4438	0·4728	0·4986	0·5217	0·5426
	·85	·3501	·3957	·4337	·4662	·4947	·5198	·5423	·5626
	·90	·3813	·4260	·4631	·4947	·5223	·5466	·5683	·5878
	·95	·4288	·4718	·5072	·5373	·5633	·5862	·6066	·6249
	·99	·5201	·5586	·5900	·6164	·6392	·6590	·6766	·6923
26	0·80	0·3046	0·3490	0·3864	0·4187	0·4473	0·4727	0·4957	0·5165
	·85	·3276	·3716	·4086	·4404	·4685	·4934	·5158	·5361
	·90	·3574	·4007	·4370	·4681	·4954	·5196	·5413	·5610
	·95	·4030	·4450	·4798	·5096	·5356	·5586	·5792	·5977
	·99	·4914	·5296	·5610	·5876	·6107	·6309	·6490	·6651
28	0·80	0·2859	0·3286	0·3648	0·3963	0·4243	0·4493	0·4720	0·4926
	·85	·3078	·3503	·3861	·4173	·4448	·4694	·4917	·5119
	·90	·3363	·3783	·4136	·4441	·4710	·4951	·5167	·5363
	·95	·3801	·4210	·4552	·4846	·5104	·5333	·5539	·5726
	·99	·4656	·5033	·5345	·5612	·5844	·6049	·6232	·6398
30	0·80	0·2694	0·3104	0·3455	0·3761	0·4035	0·4280	0·4504	0·4708
	·85	·2903	·3312	·3660	·3964	·4234	·4476	·4696	·4897
	·90	·3175	·3581	·3925	·4224	·4489	·4726	·4941	·5136
	·95	·3595	·3993	·4328	·4618	·4873	·5101	·5307	·5493
	·99	·4423	·4794	·5103	·5369	·5601	·5808	·5993	·6160
32	0·80	0·2546	0·2941	0·3280	0·3579	0·3845	0·4087	0·4307	0·4508
	·85	·2746	·3141	·3479	·3775	·4039	·4277	·4494	·4693
	·90	·3007	·3400	·3734	·4027	·4287	·4521	·4733	·4927
	·95	·3411	·3798	·4125	·4410	·4661	·4887	·5092	·5278
	·99	·4211	·4575	·4881	·5144	·5376	·5583	·5769	·5938
34	0·80	0·2414	0·2795	0·3123	0·3413	0·3673	0·3909	0·4125	0·4324
	·85	·2605	·2986	·3314	·3602	·3861	·4095	·4308	·4504
	·90	·2855	·3236	·3561	·3847	·4102	·4332	·4542	·4734
	·95	·3244	·3620	·3940	·4219	·4467	·4690	·4893	·5079
	·99	·4018	·4375	·4676	·4937	·5168	·5374	·5561	·5730
36	0·80	0·2295	0·2662	0·2980	0·3261	0·3515	0·3746	0·3958	0·4154
	·85	·2478	·2846	·3164	·3445	·3697	·3927	·4137	·4330
	·90	·2718	·3087	·3403	·3682	·3932	·4158	·4364	·4555
	·95	·3093	·3458	·3770	·4043	·4287	·4508	·4709	·4893
	·99	·3842	·4191	·4488	·4746	·4974	·5180	·5366	·5536
38	0·80	0·2187	0·2541	0·2849	0·3123	0·3370	0·3596	0·3804	0·3997
	·85	·2363	·2719	·3027	·3300	·3547	·3772	·3978	·4169
	·90	·2594	·2951	·3258	·3530	·3775	·3997	·4200	·4388
	·95	·2955	·3309	·3614	·3882	·4122	·4339	·4537	·4720
	·99	·3680	·4022	·4313	·4568	·4794	·4998	·5183	·5353
40	0·80	0·2088	0·2430	0·2729	0·2995	0·3237	0·3458	0·3662	0·3851
	·85	·2258	·2602	·2901	·3167	·3408	·3628	·3831	·4018
	·90	·2480	·2826	·3125	·3390	·3630	·3848	·4048	·4233
	·95	·2828	·3173	·3470	·3732	·3968	·4182	·4377	·4558
	·99	·3531	·3865	·4151	·4402	·4626	·4828	·5012	·5181
42	0·80	0·1998	0·2329	0·2619	0·2878	0·3113	0·3329	0·3529	0·3715
	·85	·2161	·2495	·2785	·3045	·3280	·3495	·3694	·3878
	·90	·2376	·2711	·3002	·3261	·3495	·3709	·3906	·4088
	·95	·2712	·3048	·3337	·3594	·3825	·4035	·4228	·4407
	·99	·3393	·3720	·4001	·4248	·4469	·4669	·4851	·5020

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1 \ ν_2		3	4	5	6	7	8	9	10
	P								
44	0·80	0·1916	0·2236	0·2517	0·2769	0·2999	0·3210	0·3406	0·3588
	·85	·2073	·2396	·2678	·2931	·3161	·3371	·3566	·3748
	·90	·2280	·2605	·2888	·3141	·3370	·3580	·3773	·3953
	·95	·2605	·2931	·3241	·3465	·3692	·3898	·4089	·4265
	·99	·3266	·3586	·3861	·4104	·4321	·4519	·4700	·4867
46	0·80	0·1840	0·2150	0·2423	0·2668	0·2892	0·3099	0·3290	0·3469
	·85	·1991	·2304	·2579	·2825	·3050	·3256	·3447	·3625
	·90	·2192	·2507	·2783	·3030	·3254	·3459	·3649	·3826
	·95	·2506	·2824	·3099	·3345	·3567	·3770	·3958	·4132
	·99	·3147	·3460	·3730	·3969	·4183	·4379	·4558	·4724
48	0·80	0·1770	0·2070	0·2335	0·2574	0·2793	0·2995	0·3183	0·3358
	·85	·1916	·2220	·2487	·2727	·2946	·3148	·3336	·3511
	·90	·2110	·2416	·2685	·2926	·3145	·3346	·3533	·3707
	·95	·2415	·2723	·2993	·3233	·3451	·3650	·3835	·4006
	·99	·3037	·3343	·3608	·3842	·4054	·4247	·4424	·4588
50	0·80	0·1704	0·1996	0·2254	0·2487	0·2700	0·2898	0·3082	0·3254
	·85	·1846	·2141	·2401	·2635	·2849	·3047	·3231	·3403
	·90	·2034	·2332	·2594	·2829	·3043	·3241	·3424	·3595
	·95	·2329	·2630	·2893	·3128	·3342	·3538	·3719	·3888
	·99	·2934	·3233	·3493	·3723	·3932	·4122	·4297	·4460
52	0·80	0·1644	0·1927	0·2178	0·2405	0·2613	0·2807	0·2987	0·3156
	·85	·1781	·2068	·2321	·2549	·2759	·2952	·3133	·3302
	·90	·1963	·2253	·2508	·2738	·2948	·3141	·3321	·3490
	·95	·2250	·2543	·2799	·3030	·3239	·3432	·3610	·3777
	·99	·2838	·3131	·3385	·3612	·3817	·4004	·4177	·4338
54	0·80	0·1588	0·1863	0·2107	0·2328	0·2532	0·2721	0·2898	0·3064
	·85	·1721	·1999	·2246	·2469	·2674	·2863	·3040	·3206
	·90	·1897	·2179	·2428	·2653	·2858	·3048	·3225	·3390
	·95	·2175	·2461	·2712	·2937	·3142	·3332	·3507	·3671
	·99	·2748	·3034	·3284	·3506	·3708	·3893	·4064	·4223
56	0·80	·1535	·1802	·2040	·2256	·2456	·2641	·2814	·2976
	·85	·1664	·1935	·2176	·2394	·2594	·2779	·2953	·3116
	·90	·1835	·2110	·2353	·2572	·2774	·2960	·3133	·3296
	·95	·2106	·2385	·2630	·2850	·3051	·3237	·3410	·3572
	·99	·2664	·2944	·3188	·3407	·3605	·3787	·3956	·4113
58	·80	0·1486	0·1746	0·1978	0·2189	0·2383	0·2565	0·2734	0·2894
	·85	·1611	·1875	·2110	·2322	·2518	·2700	·2870	·3030
	·90	·1777	·2045	·2282	·2497	·2694	·2876	·3047	·3207
	·95	·2041	·2313	·2552	·2768	·2966	·3148	·3318	·3477
	·99	·2584	·2858	·3098	·3312	·3508	·3687	·3854	·4009
60	0·80	0·1440	0·1693	0·1919	0·2125	0·2316	0·2493	0·2659	0·2816
	·85	·1561	·1819	·2047	·2255	·2447	·2625	·2792	·2950
	·90	·1723	·1984	·2216	·2426	·2619	·2798	·2965	·3122
	·95	·1979	·2245	·2479	·2691	·2884	·3064	·3231	·3387
	·99	·2509	·2777	·3012	·3223	·3415	·3592	·3757	·3910
62	0·80	0·1396	0·1643	0·1864	0·2065	0·2251	0·2425	0·2588	0·2742
	·85	·1515	·1765	·1989	·2192	·2380	·2555	·2718	·2873
	·90	·1672	·1927	·2153	·2358	·2547	·2723	·2887	·3042
	·95	·1921	·2181	·2410	·2617	·2807	·2983	·3148	·3302
	·99	·2438	·2701	·2932	·3139	·3328	·3502	·3664	·3816

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1 \ ν_2	P	3	4	5	6	7	8	9	10
64	0·80	0·1355	0·1596	0·1811	0·2008	0·2191	0·2361	0·2521	0·2672
	·85	·1471	·1715	·1933	·2132	·2316	·2488	·2648	·2800
	·90	·1624	·1873	·2094	·2295	·2480	·2652	·2814	·2966
	·95	·1867	·2120	·2345	·2548	·2734	·2907	·3069	·3221
	·99	·2371	·2629	·2855	·3058	·3245	·3416	·3576	·3726
66	0·80	0·1317	0·1552	0·1762	0·1955	0·2133	0·2300	0·2457	0·2605
	·85	·1429	·1668	·1881	·2076	·2256	·2424	·2582	·2731
	·90	·1578	·1821	·2038	·2234	·2416	·2585	·2744	·2893
	·95	·1815	·2063	·2283	·2482	·2665	·2835	·2994	·3144
	·99	·2308	·2560	·2782	·2982	·3165	·3335	·3492	·3640
68	0·80	0·1280	0·1509	0·1715	0·1904	0·2078	0·2242	0·2396	0·2542
	·85	·1390	·1623	·1831	·2022	·2198	·2363	·2518	·2665
	·90	·1535	·1773	·1984	·2177	·2355	·2521	·2677	·2824
	·95	·1767	·2009	·2224	·2419	·2599	·2766	·2923	·3070
	·99	·2248	·2495	·2713	·2910	·3090	·3256	·3412	·3558
70	0·80	0·1246	0·1470	0·1671	0·1855	0·2026	0·2187	0·2338	0·2482
	·85	·1353	·1580	·1784	·1971	·2144	·2306	·2458	·2602
	·90	·1495	·1727	·1934	·2123	·2297	·2461	·2614	·2758
	·95	·1720	·1958	·2168	·2360	·2536	·2701	·2855	·3000
	·99	·2191	·2433	·2647	·2840	·3018	·3182	·3335	·3479
72	0·80	0·1213	0·1432	0·1629	0·1809	0·1977	0·2134	0·2283	0·2424
	·85	·1317	·1540	·1740	·1922	·2092	·2251	·2400	·2542
	·90	·1456	·1683	·1886	·2071	·2242	·2403	·2553	·2696
	·95	·1677	·1909	·2115	·2303	·2477	·2638	·2790	·2932
	·99	·2136	·2374	·2584	·2774	·2949	·3111	·3262	·3404
74	0·80	0·1182	0·1396	0·1588	0·1765	0·1930	0·2084	0·2230	0·2369
	·85	·1284	·1502	·1697	·1876	·2043	·2199	·2346	·2485
	·90	·1419	·1641	·1840	·2022	·2190	·2347	·2495	·2635
	·95	·1635	·1862	·2065	·2249	·2419	·2578	·2727	·2868
	·99	·2085	·2318	·2524	·2711	·2883	·3043	·3192	·3332
76	0·80	0·1153	0·1362	0·1550	0·1724	0·1885	0·2037	0·2180	0·2316
	·85	·1252	·1465	·1657	·1832	·1995	·2149	·2293	·2430
	·90	·1384	·1602	·1797	·1975	·2140	·2294	·2440	·2578
	·95	·1595	·1818	·2017	·2198	·2365	·2521	·2668	·2807
	·99	·2036	·2264	·2467	·2651	·2820	·2977	·3124	·3263
78	0·80	0·1125	0·1329	0·1514	0·1684	0·1842	0·1991	0·2132	0·2266
	·85	·1222	·1430	·1618	·1790	·1950	·2101	·2243	·2378
	·90	·1351	·1564	·1755	·1930	·2092	·2244	·2387	·2523
	·95	·1557	·1776	·1971	·2148	·2313	·2466	·2611	·2748
	·99	·1989	·2213	·2413	·2593	·2760	·2915	·3060	·3196
80	0·80	0·1098	0·1298	0·1479	0·1646	0·1801	0·1948	0·2086	0·2218
	·85	·1193	·1397	·1581	·1750	·1907	·2055	·2195	·2327
	·90	·1320	·1528	·1715	·1887	·2046	·2196	·2337	·2470
	·95	·1521	·1735	·1927	·2101	·2263	·2414	·2556	·2691
	·99	·1944	·2164	·2360	·2538	·2702	·2855	·2998	·3133
82	0·80	0 1073	0 1269	0 1446	0·1610	0·1762	0·1906	0·2042	0·2172
	·85	·1165	·1366	·1546	·1712	·1866	·2011	·2149	·2279
	·90	·1289	·1494	·1677	·1846	·2002	·2149	·2288	·2420
	·95	·1487	·1697	·1884	·2056	·2215	·2364	·2504	·2637
	·99	·1901	·2118	·2310	·2485	·2647	·2797	·2938	·3071

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1 \ ν_2	P	3	4	5	6	7	8	9	10
84	0·80	0·1048	0·1240	0·1414	0·1575	0·1725	0·1866	0·2000	0·2127
	·85	·1139	·1335	·1512	·1675	·1827	·1969	·2105	·2233
	·90	·1260	·1461	·1641	·1807	·1960	·2105	·2241	·2371
	·95	·1454	·1660	·1844	·2013	·2169	·2316	·2454	·2584
	·99	·1860	·2073	·2262	·2434	·2594	·2742	·2881	·3013
86	0·80	0·1025	0·1213	0·1384	0·1541	0·1689	0·1828	0·1959	0·2085
	·85	·1114	·1306	·1480	·1640	·1789	·1929	·2062	·2189
	·90	·1233	·1429	·1606	·1769	·1920	·2062	·2197	·2324
	·95	·1422	·1625	·1806	·1971	·2125	·2269	·2405	·2534
	·99	·1821	·2030	·2216	·2386	·2542	·2689	·2826	·2956
88	0·80	0·1003	0·1188	0·1355	0·1510	0·1654	0·1791	0·1920	0·2044
	·85	·1090	·1279	·1449	·1606	·1753	·1891	·2021	·2146
	·90	·1206	·1399	·1573	·1733	·1881	·2021	·2154	·2279
	·95	·1392	·1591	·1769	·1932	·2083	·2225	·2359	·2486
	·99	·1783	·1989	·2172	·2339	·2493	·2637	·2773	·2901
90	0·80	0·0981	0·1163	0·1327	0·1479	0·1621	0·1755	0·1883	0·2004
	·85	·1067	·1252	·1419	·1574	·1718	·1854	·1982	·2105
	·90	·1181	·1371	·1541	·1698	·1844	·1982	·2112	·2236
	·95	·1363	·1558	·1733	·1893	·2042	·2182	·2314	·2440
	·99	·1747	·1949	·2129	·2294	·2446	·2588	·2722	·2848
92	0·80	0·0961	0·1139	0·1300	0·1450	0·1589	0·1721	0·1847	0·1967
	·85	·1045	·1227	·1391	·1542	·1684	·1818	·1945	·2066
	·90	·1157	·1343	·1510	·1665	·1809	·1944	·2072	·2195
	·95	·1336	·1527	·1699	·1857	·2003	·2141	·2271	·2395
	·99	·1712	·1911	·2088	·2250	·2400	·2541	·2673	·2798
94	0·80	0·0942	0·1116	0·1275	0·1421	0·1559	0·1689	0·1812	0·1930
	·85	·1024	·1202	·1363	·1513	·1652	·1784	·1908	·2028
	·90	·1134	·1316	·1481	·1633	·1774	·1908	·2034	·2154
	·95	·1309	·1497	·1666	·1821	·1966	·2101	·2230	·2352
	·99	·1679	·1874	·2049	·2209	·2356	·2495	·2625	·2748
96	0·80	0·0923	0·1094	0·1250	0·1394	0·1529	0·1657	0·1779	0·1895
	·85	·1003	·1179	·1337	·1484	·1621	·1750	·1873	·1991
	·90	·1111	·1291	·1453	·1602	·1741	·1873	·1997	·2116
	·95	·1283	·1468	·1635	·1787	·1930	·2063	·2190	·2310
	·99	·1647	·1839	·2011	·2168	·2314	·2451	·2579	·2701
98	0·80	0·0905	0·1073	0·1226	0·1368	0·1501	0·1627	0·1747	0·1861
	·85	·0984	·1156	·1312	·1456	·1591	·1719	·1840	·1955
	·90	·1090	·1266	·1425	·1572	·1709	·1839	·1962	·2079
	·95	·1259	·1441	·1604	·1755	·1895	·2026	·2151	·2270
	·99	·1616	·1805	·1975	·2130	·2273	·2408	·2535	·2655
100	0·80	0·0887	0·1053	0·1203	0·1343	0·1474	0·1598	0·1716	0·1828
	·85	·0965	·1134	·1288	·1429	·1562	·1688	·1807	·1921
	·90	·1069	·1242	·1399	·1543	·1679	·1806	·1927	·2043
	·95	·1235	·1414	·1575	·1723	·1861	·1991	·2114	·2231
	·99	·1586	·1772	·1939	·2092	·2234	·2367	·2492	·2611
102	0·80	0·0871	0·1033	0·1181	0·1318	0·1447	0·1569	0·1686	0·1797
	·85	·0947	·1113	·1264	·1404	·1534	·1658	·1776	·1888
	·90	·1049	·1219	·1374	·1516	·1649	·1775	·1894	·2008
	·95	·1212	·1388	·1547	·1692	·1829	·1957	·2078	·2193
	·99	·1557	·1741	·1905	·2056	·2196	·2327	·2451	·2568

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1 \ ν_2	P	3	4	5	6	7	8	9	10
104	0·80	0·0855	0·1014	0·1160	0·1295	0·1422	0·1542	0·1657	0·1766
	·85	·0930	·1093	·1241	·1379	·1508	·1630	·1746	·1856
	·90	·1030	·1197	·1349	·1489	·1620	·1744	·1862	·1974
	·95	·1190	·1363	·1519	·1663	·1797	·1923	·2043	·2157
	·99	·1529	·1710	·1872	·2021	·2159	·2288	·2411	·2527
106	0·80	0·0839	0·0996	0·1139	0·1272	0·1397	0·1516	0·1629	0·1737
	·85	·0913	·1074	·1220	·1355	·1482	·1602	·1716	·1826
	·90	·1011	·1176	·1326	·1464	·1593	·1715	·1831	·1942
	·95	·1169	·1340	·1493	·1635	·1767	·1891	·2009	·2122
	·99	·1503	·1681	·1841	·1987	·2123	·2251	·2372	·2487
108	0·80	0·0824	0·0979	0·1120	0·1251	0·1374	0·1490	0·1602	0·1708
	·85	·0897	·1055	·1198	·1332	·1457	·1575	·1688	·1796
	·90	·0993	·1156	·1303	·1439	·1566	·1686	·1801	·1910
	·95	·1149	·1316	·1468	·1607	·1737	·1860	·1977	·2088
	·99	·1477	·1652	·1810	·1954	·2089	·2215	·2334	·2448
110	0·80	0·0810	0·0962	0·1100	0·1229	0·1351	0·1466	0·1576	0·1681
	85	·0881	·1037	·1178	·1309	·1433	·1549	·1661	·1767
	·90	·0976	·1136	·1281	·1415	·1540	·1659	·1772	·1880
	·95	·1129	·1294	·1443	·1581	·1709	·1830	·1945	·2055
	·99	·1452	·1625	·1780	·1923	·2056	·2180	·2298	·2410
112	0·80	0·0796	0·0945	0·1082	0·1209	0·1329	0·1442	0·1550	0·1654
	·85	·0866	·1019	·1158	·1288	·1409	·1524	·1634	·1739
	·90	·0960	·1117	·1259	·1391	·1515	·1632	·1744	·1850
	·95	·1110	·1273	·1419	·1555	·1682	·1801	·1915	·2023
	·99	·1428	·1598	·1752	·1892	·2023	·2146	·2263	·2373
114	0·80	0·0782	0·0930	0·1064	0·1189	0·1307	0·1419	0·1526	0·1628
	·85	·0851	·1002	·1139	·1267	·1387	·1500	·1608	·1712
	·90	·0944	·1098	·1239	·1369	·1491	·1606	·1716	·1822
	·95	·1091	·1252	·1396	·1530	·1655	·1773	·1885	·1992
	·99	·1405	·1573	·1724	·1863	·1992	·2113	·2228	·2338
116	0·80	0·0769	0·0914	0·1047	0·1170	0·1286	0·1397	0·1502	0·1603
	·85	·0837	·0986	·1121	·1247	·1365	·1477	·1583	·1686
	·90	·0928	·1080	·1219	·1347	·1468	·1581	·1690	·1794
	·95	·1074	·1231	·1374	·1506	·1629	·1746	·1856	·1962
	·99	·1382	·1548	·1697	·1834	·1961	·2082	·2195	·2303
118	0·80	0·0757	0·0900	0·1030	0·1152	0·1266	0·1375	0·1479	0·1579
	·85	·0824	·0970	·1103	·1227	·1343	·1454	·1559	·1660
	·90	·0913	·1063	·1200	·1326	·1445	·1557	·1664	·1767
	·95	·1056	·1212	·1353	·1483	·1604	·1719	·1829	·1933
	·99	·1360	·1524	·1671	·1806	·1932	·2051	·2163	·2270
120	0·80	0·0744	0·0885	0·1014	0·1134	0·1247	0·1354	0·1457	0·1555
	·85	·0810	·0954	·1086	·1208	·1323	·1432	·1536	·1636
	·90	·0898	·1046	·1181	·1306	·1423	·1534	·1640	·1741
	·95	·1040	·1193	·1332	·1460	·1580	·1694	·1802	·1905
	·99	·1339	·1500	·1645	·1779	·1903	·2021	·2132	·2237
122	0·80	0·0733	0·0871	0·0998	0·1116	0·1228	0·1334	0·1435	0·1532
	·85	·0798	·0940	·1069	·1189	·1303	·1410	·1513	·1612
	·90	·0884	·1030	·1163	·1286	·1401	·1511	·1615	·1715
	·95	·1023	·1175	·1311	·1438	·1557	·1669	·1775	·1877
	·99	·1319	·1478	·1621	·1753	·1876	·1991	·2101	·2206

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1	ν_2 / P	3	4	5	6	7	8	9	10
124	0·80	0·0721	0·0858	0·0983	0·1100	0·1209	0·1314	0·1414	0·1510
	·85	·0785	·0925	·1053	·1172	·1283	·1390	·1491	·1588
	·90	·0871	·1014	·1145	·1267	·1381	·1489	·1592	·1691
	·95	·1008	·1157	·1292	·1417	·1534	·1644	·1750	·1851
	·99	·1299	·1456	·1597	·1727	·1849	·1963	·2072	·2175
126	0·80	0·0710	0·0845	0·0968	0·1083	0·1192	0·1295	0·1394	0·1488
	·85	·0773	·0911	·1037	·1154	·1265	·1369	·1470	·1566
	·90	·0857	·0999	·1128	·1248	·1361	·1467	·1569	·1667
	·95	·0992	·1139	·1273	·1396	·1512	·1621	·1725	·1825
	·99	·1279	·1434	·1574	·1702	·1822	·1936	·2043	·2145
128	0·80	0·0699	0·0832	0·0954	0·1067	0·1174	0·1276	0·1374	0·1467
	·85	·0761	·0898	·1022	·1137	·1246	·1350	·1449	·1544
	·90	·0844	·0984	·1112	·1230	·1341	·1446	·1547	·1644
	·95	·0978	·1123	·1254	·1376	·1490	·1598	·1701	·1799
	·99	·1261	·1413	·1551	·1678	·1797	·1909	·2015	·2116
130	0·80	0·0689	0·0820	0·0940	0·1052	0·1158	0·1258	0·1354	0·1447
	·85	·0750	·0884	·1007	·1121	·1229	·1331	·1429	·1522
	·90	·0832	·0970	·1096	·1212	·1322	·1426	·1526	·1621
	·95	·0963	·1106	·1236	·1356	·1469	·1576	·1678	·1775
	·99	·1242	·1393	·1529	·1655	·1772	·1883	·1988	·2088
132	0·80	0·0679	0·0808	0·0926	0·1037	0·1141	0·1241	0·1336	0·1427
	·85	·0739	·0872	·0992	·1105	·1211	·1312	·1409	·1502
	·90	·0820	·0956	·1080	·1195	·1304	·1406	·1505	·1599
	·95	·0949	·1091	·1219	·1337	·1449	·1554	·1655	·1751
	·99	·1225	·1374	·1508	·1632	·1748	·1857	·1961	·2060
134	0·80	0·0669	0·0797	0·0913	0·1022	0·1125	0·1223	0·1317	0·1408
	·85	·0728	·0859	·0978	·1090	·1195	·1294	·1390	·1481
	·90	·0808	·0942	·1065	·1179	·1286	·1387	·1484	·1578
	·95	·0936	·1075	·1202	·1319	·1429	·1533	·1633	·1728
	·99	·1207	·1355	·1488	·1610	·1725	·1833	·1936	·2034
136	0·80	0·0660	0·0785	0·0901	0·1008	0·1110	0·1207	0·1300	0·1389
	·85	·0718	·0847	·0965	·1075	·1178	·1277	·1371	·1462
	·90	·0797	·0929	·1050	·1162	·1268	·1369	·1465	·1557
	·95	·0923	·1060	·1185	·1301	·1410	·1513	·1611	·1705
	·99	·1191	·1336	·1467	·1589	·1702	·1809	·1910	·2007
138	0·80	0·0650	0·0774	0·0888	0·0994	0·1095	0·1191	0·1282	0·1371
	·85	·0708	·0835	·0952	·1060	·1162	·1260	·1353	·1442
	·90	·0785	·0916	·1036	·1147	·1251	·1350	·1445	·1536
	·95	·0910	·1046	·1169	·1284	·1391	·1493	·1590	·1683
	·99	·1174	·1318	·1448	·1568	·1680	·1785	·1886	·1982
140	0·80	0·0641	0·0764	0·0876	0·0981	0·1080	0·1175	0·1266	0·1353
	·85	·0698	·0824	·0939	·1046	·1147	·1243	·1335	·1424
	·90	·0775	·0904	·1022	·1131	·1235	·1333	·1427	·1517
	·95	·0897	·1032	·1153	·1267	·1373	·1473	·1569	·1662
	·99	·1159	·1301	·1429	·1547	·1658	·1763	·1862	·1957
142	0·80	0·0632	0·0753	0·0864	0·0968	0·1066	0·1160	0·1249	0·1336
	·85	·0689	·0813	·0926	·1032	·1132	·1227	·1318	·1406
	·90	·0764	·0892	·1008	·1116	·1218	·1315	·1408	·1497
	·95	·0885	·1018	·1138	·1250	·1355	·1455	·1550	·1641
	·99	·1143	·1283	·1410	·1527	·1637	·1740	·1839	·1933

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1 \ ν_2		3	4	5	6	7	8	9	10
	P								
144	0·80	0·0624	0·0743	0·0853	0·0955	0·1052	0·1145	0·1233	0·1319
	·85	·0679	·0802	·0914	·1018	·1117	·1211	·1301	·1388
	·90	·0754	·0880	·0995	·1102	·1203	·1299	·1390	·1478
	·95	·0874	·1004	·1123	·1234	·1338	·1436	·1530	·1620
	·99	·1128	·1267	·1392	·1508	·1616	·1719	·1816	·1909
146	0·80	0·0616	0·0734	0·0842	0·0943	0·1039	0·1130	0·1218	0·1302
	·85	·0670	·0791	·0902	·1005	·1103	·1196	·1285	·1371
	·90	·0744	·0868	·0982	·1088	·1187	·1282	·1373	·1460
	·95	·0862	·0991	·1109	·1218	·1321	·1418	·1511	·1600
	·99	·1114	·1251	·1374	·1489	·1596	·1697	·1794	·1886
148	0·80	0·0608	0·0724	0·0831	0·0931	0·1026	0·1116	0·1203	0·1286
	·85	·0662	·0781	·0890	·0992	·1089	·1181	·1269	·1354
	·90	·0734	·0857	·0969	·1074	·1173	·1266	·1356	·1442
	·95	·0851	·0979	·1095	·1203	·1304	·1401	·1493	·1581
	·99	·1099	·1235	·1357	·1470	·1577	·1677	·1772	·1864
150	0·80	0·0600	0·0715	0·0820	0·0919	0·1013	0·1102	0·1188	0·1270
	·85	·0653	·0771	·0879	·0980	·1075	·1166	·1253	·1337
	·90	·0725	·0846	·0957	·1060	·1158	·1251	·1339	·1425
	·95	·0840	·0966	·1081	·1188	·1288	·1383	·1474	·1562
	·99	·1085	·1219	·1340	·1452	·1557	·1657	·1751	·1842
152	0·80	0·0592	0·0706	0·0810	0·0908	0·1000	0·1088	0·1173	0·1255
	·85	·0645	·0761	·0868	·0968	·1062	·1152	·1238	·1321
	·90	·0716	·0836	·0945	·1047	·1144	·1235	·1323	·1408
	·95	·0829	·0954	·1068	·1173	·1273	·1367	·1457	·1543
	·99	·1072	·1204	·1324	·1435	·1539	·1637	·1731	·1820
154	0·80	0·0584	0·0697	0·0800	0·0896	0·0988	0·1075	0·1159	0·1240
	·85	·0637	·0752	·0857	·0956	·1049	·1138	·1223	·1305
	·90	·0707	·0825	·0934	·1035	·1130	·1221	·1308	·1391
	·95	·0819	·0942	·1055	·1159	·1257	·1350	·1440	·1525
	·99	·1059	·1190	·1308	·1418	·1521	·1618	·1710	·1799
156	0·80	0·0577	0 0688	0·0790	0·0886	0·0976	0·1062	0·1145	0·1225
	·85	·0629	·0743	·0847	·0944	·1037	·1125	·1209	·1290
	·90	·0698	·0815	·0922	·1022	·1116	·1206	·1292	·1375
	·95	·0809	·0931	·1042	·1145	·1242	·1335	·1423	·1507
	·99	·1046	·1175	·1293	·1401	·1503	·1599	·1691	·1779
158	0·80	0·0570	0·0680	0·0781	0·0875	0·0964	0·1050	0·1132	0·1211
	·85	·0621	·0734	·0837	·0933	·1024	·1111	·1195	·1275
	·90	·0689	·0805	·0911	·1010	·1103	·1192	·1277	·1359
	·95	·0799	·0920	·1029	·1132	·1228	·1319	·1406	·1490
	·99	·1033	·1161	·1277	·1385	·1485	·1581	·1672	·1759
160	0·80	0·0563	0·0672	0·0771	0·0865	0·0953	0·1038	0·1119	0·1197
	·85	·0613	·0725	·0827	·0922	·1012	·1098	·1181	·1260
	·90	·0681	·0796	·0900	·0998	·1090	·1178	·1262	·1343
	·95	·0789	·0909	·1017	·1118	·1213	·1304	·1390	·1473
	·99	·1021	·1148	·1262	·1369	·1468	·1563	·1653	·1739
162	0·80	0·0556	0·0664	0·0762	0·0854	0·0942	0·1026	0·1106	0·1184
	·85	·0606	·0716	·0817	·0911	·1000	·1086	·1167	·1246
	·90	·0673	·0786	·0890	·0986	·1078	·1165	·1248	·1328
	·95	·0780	·0898	·1005	·1105	·1200	·1289	·1375	·1457
	·99	·1009	·1134	·1248	·1353	·1452	·1545	·1634	·1720

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1	ν_2 / P	3	4	5	6	7	8	9	10
164	0·80	0·0550	0·0656	0·0753	0·0845	0·0931	0·1014	0·1093	0·1170
	·85	·0599	·0708	·0807	·0901	·0989	·1073	·1154	·1232
	·90	·0665	·0777	·0879	·0975	·1065	·1151	·1234	·1313
	·95	·0771	·0888	·0994	·1093	·1186	·1275	·1359	·1441
	·99	·0997	·1121	·1234	·1338	·1436	·1528	·1616	·1701
166	0·80	0·0543	0·0648	0·0744	0·0835	0·0921	0·1002	0·1081	0·1157
	·85	·0592	·0699	·0798	·0890	·0978	·1061	·1141	·1219
	·90	·0657	·0768	·0869	·0964	·1053	·1139	·1220	·1299
	·95	·0762	·0877	·0982	·1080	·1173	·1260	·1344	·1425
	·99	·0986	·1109	·1220	·1323	·1420	·1511	·1599	·1683
168	0·80	0·0537	0·0641	0·0736	0·0825	0·0910	0·0991	0·1069	0·1144
	·85	·0585	·0691	·0789	·0880	·0967	·1049	·1129	·1205
	·90	·0649	·0759	·0859	·0953	·1042	·1126	·1207	·1285
	·95	·0753	·0867	·0971	·1068	·1160	·1246	·1330	·1409
	·99	·0975	·1096	·1206	·1308	·1404	·1495	·1582	·1665
170	0·80	0·0531	0·0633	0·0728	0·0816	0·0900	0·0980	0·1057	0·1132
	·85	·0578	·0684	·0780	·0870	·0956	·1038	·1116	·1192
	·90	·0642	·0751	·0850	·0942	·1030	·1114	·1194	·1271
	·95	·0745	·0858	·0961	·1056	·1147	·1233	·1315	·1394
	·99	·0964	·1084	·1193	·1294	·1389	·1479	·1565	·1647
172	0·80	0·0525	0·0626	0·0720	0·0807	0·0890	0·0970	0·1046	0·1120
	·85	·0572	·0676	·0771	·0861	·0946	·1027	·1104	·1179
	·90	·0635	·0742	·0840	·0932	·1019	·1102	·1181	·1257
	·95	·0736	·0848	·0950	·1045	·1134	·1220	·1301	·1380
	·99	·0953	·1072	·1180	·1280	·1374	·1463	·1548	·1630
174	0·80	0·0519	0·0619	0·0712	0·0798	0·0880	·0959	·1035	·1108
	·85	·0565	·0668	·0763	·0851	·0935	·1016	·1092	·1167
	·90	·0628	·0734	·0831	·0922	·1008	·1090	·1168	·1244
	·95	·0728	·0839	·0940	·1034	·1122	·1207	·1287	·1365
	·99	·0943	·1060	·1167	·1266	·1360	·1448	·1532	·1613
176	0·80	0·0513	0·0612	0·0704	0·0790	0·0871	0·0949	0·1024	0·1096
	·85	·0559	·0661	·0755	·0842	·0925	·1005	·1081	·1154
	·90	·0621	·0726	·0822	·0912	·0997	·1078	·1156	·1231
	·95	·0720	·0830	·0929	·1023	·1110	·1194	·1274	·1351
	·99	·0932	·1049	·1155	·1253	·1345	·1433	·1516	·1597
178	0·80	0·0507	0·0606	0·0696	0·0781	0·0862	0·0939	0·1013	0·1085
	·85	·0553	·0654	·0746	·0833	·0915	·0994	·1070	·1142
	·90	·0614	·0718	·0813	·0902	·0987	·1067	·1144	·1218
	·95	·0712	·0821	·0920	·1012	·1099	·1181	·1261	·1337
	·99	·0922	·1038	·1143	·1240	·1331	·1418	·1501	·1580
180	0·80	0·0502	0·0599	0·0689	0·0773	0·0853	0·0929	0·1003	0·1074
	·85	·0547	·0647	·0739	·0824	·0906	·0984	·1059	·1131
	·90	·0607	·0710	·0805	·0893	·0976	·1056	·1132	·1206
	·95	·0705	·0812	·0910	·1001	·1087	·1169	·1248	·1323
	·99	·0913	·1027	·1131	·1227	·1318	·1404	·1486	·1565
182	0·80	0·0497	0·0593	0·0681	0·0765	0·0844	0·0919	0·0992	0·1063
	·85	·0541	·0640	·0731	·0816	·0896	·0974	·1048	·1119
	·90	·0601	·0703	·0796	·0883	·0966	·1045	·1121	·1194
	·95	·0697	·0803	·0900	·0991	·1076	·1157	·1235	·1310
	·99	·0903	·1016	·1119	·1215	·1304	·1390	·1471	·1549

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

Generalized Beta distribution (cont.)

ν_1 \ ν_2		3	4	5	6	7	8	9	10
	P								
184	0·80	·0491	·0587	·0674	·0757	·0835	·0910	·0982	·1052
	·85	·0535	·0633	·0723	·0807	·0887	·0964	·1037	·1108
	·90	·0595	·0695	·0788	·0874	·0956	·1034	·1109	·1182
	·95	·0690	·0795	·0891	·0981	·1065	·1146	·1223	·1297
	·99	·0894	·1006	·1108	·1202	·1291	·1376	·1456	·1534
186	0·80	·0486	·0580	·0667	·0749	·0826	·0901	·0972	·1041
	·85	·0530	·0627	·0716	·0799	·0878	·0954	·1027	·1097
	·90	·0588	·0688	·0780	·0865	·0946	·1024	·1098	·1170
	·95	·0683	·0787	·0882	·0971	·1054	·1134	·1211	·1284
	·99	·0884	·0995	·1096	·1190	·1278	·1362	·1442	·1519
188	0·80	0·0481	0·0574	0·0661	0·0741	0·0818	0·0892	0·0962	0·1031
	·85	·0524	·0620	·0708	·0791	·0869	·0944	·1016	·1086
	·90	·0582	·0681	·0772	·0857	·0937	·1014	·1087	·1158
	·95	·0676	·0779	·0873	·0961	·1044	·1123	·1199	·1272
	·99	·0875	·0985	·1085	·1178	·1266	·1349	·1428	·1504
190	0·80	0·0476	0·0569	0·0654	0·0734	0·0810	0·0883	0·0953	0·1021
	·85	·0519	·0614	·0701	·0783	·0861	·0935	·1006	·1075
	·90	·0576	·0674	·0764	·0848	·0928	·1004	·1077	·1147
	·95	·0669	·0771	·0864	·0951	·1033	·1112	·1187	·1259
	·99	·0867	·0976	·1075	·1167	·1253	·1336	·1414	·1490
192	0·80	0·0471	0·0563	0·0647	0·0727	0·0802	0·0874	0·0944	0·1011
	·85	·0514	·0608	·0694	·0775	·0852	·0926	·0997	·1065
	·90	·0570	·0667	·0756	·0840	·0918	·0994	·1066	·1136
	·95	·0662	·0763	·0856	·0942	·1023	·1101	·1175	·1247
	·99	·0858	·0966	·1064	·1155	·1241	·1323	·1401	·1476
194	0·80	0·0467	0·0557	0·0641	0·0719	0·0794	0·0866	0·0935	0·1001
	·85	·0508	·0602	·0687	·0768	·0844	·0917	·0987	·1055
	·90	·0565	·0661	·0749	·0831	·0910	·0984	·1056	·1125
	·95	·0655	·0756	·0847	·0933	·1013	·1090	·1164	·1235
	·99	·0849	·0956	·1054	·1144	·1229	·1310	·1388	·1462

This table gives the values of x for which $\Pr(\theta_{\max.} < x) = I_x(3; p, q) = P$, where $p = \frac{1}{2}(\nu_2 - 2)$, $q = \frac{1}{2}(\nu_1 - 2)$.

STATISTICAL ANALYSIS USING LOCAL PROPERTIES OF SMOOTHLY HETEROMORPHIC STOCHASTIC SERIES

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SUMMARY. A widening of the concept of stationarity leads to the concepts of smooth heteromorphy and local homomorphy in stochastic series, obviating much of the need for the introduction of trend into structural specifications of statistical data. It is shown that formulae for sampling properties of local statistics in stationary stochastic series are applicable as they stand to series having these less restricted properties.

1. Introduction

In a recent paper (1955*a*) the author established the principle that the sampling properties, and notably the standard errors, of statistics constructed from *local* comparisons of terms in stationary normal stochastic series were approximately deducible from the short term variational properties of the series themselves. In making practical use of such statistics, it is not necessary to know the mean, variance, or serial correlations of the series, all of which are dependent on the long-term variational properties of the series; it suffices to know the values of the serial and lag variation functions for short lags, i.e. the mean semi-squared differences of terms in the series separated by distances which are short.

This principle is important because in many practical applications the stretches of series which constitute the data are much too short to provide accurate and unbiased estimates of long-term variational properties, whereas accurate and unbiased estimates of the lag variation functions for short lags can be obtained even, say, by combining evidence from short scraps of series having the same variational properties but different means.

In recent papers the author has proposed the application of this principle in a number of techniques, including the following:

(i) Linear (1952; Hebden & Jowett, 1952) and spatial (1955*a*) systematic sampling.

(ii) Trend-reduced regression analysis (1955*b*).

(iii) Accuracy of serial variation statistics (1955*a*) and the fitting of serial variation curves (Davies & Jowett, 1956).

(iv) Systematic linear experimental arrangements, and the comparison of cycle phase means and interpenetrating sample means (1955*c*).

(v) The significance of changes in level between successive stretches from the same time series (Jump analysis) (1955*d*).

In these applications, the long-term variational properties of the series are largely irrelevant, since long-term serial variation parameters are effectively differenced away by operations involved in the sampling formulae. The question therefore arises of whether for application of these techniques and formulae the long-term properties need in fact be stationary at all, since their parameters are not considered except in a general sort of way, in making an assumption of smoothness of change with increasing lags. It will be shown in this paper that so long as certain smoothness assumptions hold, these long-term parameters need not be invariant under translation for the formulae to hold (and without modification). Furthermore, these smoothness assumptions lead to the definition of a more general class of stochastic series which may be termed *smoothly heteromorphic*, and, in so far as the local

parameters of such series are approximately invariant under translation, *locally homomorphic*. These terms are new, and the second is proposed as being more acceptable on semantic grounds than the tautological term *locally stationary* which is suggested by current terminology. It is also useful to use the unqualified concept of *homomorphy* (contr. *heteromorphy*) as implying effective invariance of a parameter under translation between certain limits only, not under all translations as in stationary series.

These new definitions are important from the theoretical point of view; smoothly heteromorphic and locally homomorphic series may be incorporated as error terms into models for data involving successively recorded elements without the embarrassing theoretical implications (e.g. of indefinite potential continuation) implicit in assuming stationary error terms; furthermore, they may be used in many circumstances where the assumption of strict invariance under translation would clearly not be justified. Their practical importance lies mainly in permitting the extended use of formulae already established for stationary series in (i)–(v), an extended use for which the formulae require little if any modification. Hence, when the occasion for practical use of these formulae arises, the question of whether the series involved are strictly stationary often need not even be considered.

In establishing this as a general principle, the exposition which forms the nucleus of this paper has necessarily become rather generalized and abstract; yet since the way in which the principle works is fundamentally simple, it will be demonstrated first in relation to a particular formula, taken from (iv) and used for a similar purpose in (1955*a*).

2. Development of ideas in relation to a specific example

Suppose $x(t)$ to be a random function of t, such as a statistical time series variate for continuous time t. For any particular value t_α of t, $x(t_\alpha)$ will be taken to have a probability distribution with mean $\mu(t_\alpha)$ and standard deviation $\sigma(t_\alpha)$; in general these parameters will not be constant for all values of t_α. For any pair of values t_α, t_β, the covariance $\operatorname{cov}[x(t_\alpha), x(t_\beta)]$ will be a function of both t_α and t_β, not merely, as in stationary series, of their difference, and will be related to the serial variation function

$$\delta(t_\alpha, t_\beta) = E[\tfrac{1}{2}(x(t_\alpha) - x(t_\beta))^2], \tag{1}$$

which is more useful than the covariance in practical work, by the formula

$$\delta(t_\alpha, t_\beta) = \tfrac{1}{2}(\mu(t_\alpha) - \mu(t_\beta))^2 + \tfrac{1}{2}\sigma^2(t_\alpha) + \tfrac{1}{2}\sigma^2(t_\beta) - \operatorname{cov}[x(t_\alpha), x(t_\beta)]. \tag{2}$$

A simple statistic U, which might be used to test the significance of the difference between two phase means, at interval θw, $(0 < \theta \leqslant \frac{1}{2})$, of a suspected cycle of period w, is given by the formula

$$nU = [x(T) - x(T+\theta w)] + [x(T+w) - x(T+w+\theta w)] + \ldots + [x(T+\overline{n-1}w) - x(T+\overline{n-1}w+\theta w)]. \tag{3}$$

The sampling variance of U is given by

$$n^2 \operatorname{var} U = \sum_{\alpha,\beta=0}^{n-1} \operatorname{cov}\{[x(T+\alpha w) - x(T+\alpha w+\theta w)], [x(T+\beta w) - x(T+\beta w+\theta w)]\}, \tag{4}$$

$$= \sum_{\alpha,\beta=0}^{n-1} \{\operatorname{cov}[x(T+\alpha w), x(T+\beta w)] + \operatorname{cov}[x(T+\alpha w+\theta w), x(T+\beta w+\theta w)] - \operatorname{cov}[x(T+\alpha w), x(T+\beta w+\theta w)] - \operatorname{cov}[x(T+\alpha w+\theta w), x(T+\beta w)]\}, \tag{5}$$

$$= \sum_{\alpha,\beta=0}^{n-1} \tfrac{1}{2}\{[\mu(T+\alpha w)-\mu(T+\beta w)]^2+[\mu(T+\alpha w+\theta w)-\mu(T+\beta w+\theta w)]^2$$
$$-[\mu(T+\alpha w)-\mu(T+\beta w+\theta w)]^2-[\mu(T+\alpha w+\theta w)-\mu(T+\beta w)]^2\}$$
$$-\delta(T+\alpha w, T+\beta w)-\delta(T+\alpha w+\theta w, T+\beta w+\theta w)$$
$$+\delta(T+\alpha w, T+\beta w+\theta w)+\delta(T+\alpha w+\theta w, T+\beta w), \tag{6}$$

$$= \sum_{\alpha,\beta=0}^{n-1} \{-[\mu(T+\alpha w)+\mu(T+\alpha w+\theta w)][\mu(T+\beta w)-\mu(T+\beta w+\theta w)]$$
$$-\delta(T+\alpha w, T+\beta w)-\delta(T+\alpha w+\theta w, T+\beta w+\theta w)$$
$$+\delta(T+\alpha w, T+\beta w+\theta w)+\delta(T+\alpha w+\theta w, T+\beta w)\}. \tag{7}$$

If $x(t)$ were stationary, with serial variation function $\delta^*(\tau)$, this would reduce to

$$n^2 \operatorname{var} U = \sum_{\alpha,\beta=0}^{n-1} [\delta^*(\overline{\alpha-\beta-\theta w})-2\delta^*(\overline{\alpha-\beta w})+\delta^*(\overline{\alpha-\beta+\theta w})]. \tag{8}$$

Each of the terms of the summation in (8) is a second difference and will be small if $\delta^*(\tau)$ is approximately linear in τ over the interval $|\tau-\overline{\alpha-\beta w}| \leqslant \theta w$, which in practice happens for large, moderate and sometimes even quite small values of $|\alpha-\beta|$. The number of terms of the summation which are non-negligible is then of order n, not n^2, and these effectively involve values of $\delta^*(\tau)$ only when τ is small compared with nw; evidently, then, the sampling variance of U, a statistic made up of the short-term comparisons in square brackets in (8), is dependent only on local variational properties of $x(t)$. The questions now arise of how far this remains true if the assumption of stationarity is discarded, and of what restrictions of an acceptably realistic kind have to be placed on the probability parameters of $x(t)$ in order that formulae such as (7) should themselves admit of simplified expression in terms of such local variational properties.

In answering these questions we are led to the concepts of *local homomorphy* and *smooth heteromorphy*. Local homomorphy implies that if we examine the local behaviour of a series, i.e. the probability structure of its random variation in short stretches, we shall find that it has much the same form from one part of the series to another; smooth heteromorphy implies that changes in the form of its structure, even for wider behaviour in longer stretches, occur gradually and smoothly as we move along the series. These are concepts which may be applied to particular parameters of the probability structure. For example, the function $\delta(t_\alpha, t_\beta)$ may be represented geometrically by a serial variation surface. Since the function is unchanged by the inversion of its arguments, and is zero when they are equal, it will take the form of two sheets which rise symmetrically from a cuspidal edge (or less commonly a line of ordinary minima) along the line $t_\alpha = t_\beta$ in the (t_α, t_β)-plane. For series heteromorphic in $\delta(t_\alpha, t_\beta)$ the cross-section of the surface in any specified direction perpendicular to the (t_α, t_β)-plane changes as its point of intersection moves along the cuspidal edge; smooth heteromorphy implies that the change is smooth in some sense, local homomorphy that it is small in the neighbourhood of the cusp. For strictly homomorphic series the cross-section does not change at all, and this carries with it an implication of symmetry in the cross-section which does not hold in general for heteromorphic series. An illustration of some evenly spaced cross-sections of a locally homomorphic, smoothly heteromorphic serial variation surface, plotted against $t_\alpha - t_\beta$ as abscissa, is given in Fig. 1.

The differencing operation on the serial variation function in each term of the summation in (7) is such that if the function is planar the difference is zero. If the square of which the four vertices are the points

$$(T+\alpha w, T+\beta w),\ (T+\alpha w+\theta w, T+\beta w+\theta w),\ (T+\alpha w, T+\beta w+\theta w),\ (T+\alpha w+\theta w, T+\beta w)$$

does not include any part of the cuspidal edge or its neighbourhood, this difference will therefore tend to be small, and will decrease as its distance from the cuspidal edge increases, i.e. as the serial variation surface becomes increasingly planar. In the contrary case, when the square includes any part of the cuspidal edge or is near to it, the difference in (7), while not negligible, will tend to be invariant under translation, if there is local homomorphy. We are therefore justified in replacing $\delta(t_\alpha, t_\beta)$ in (7) by a function of separation $\delta^*(t_\alpha - t_\beta)$ which has a cusp at zero and a geometrical representation resembling a section of the serial variation surface such as those in Fig. 1; it could even be a stationary serial variation curve, having the same shape at and near the cusp and similar characteristics of smoothness away from the cusp. The differencing operations in (7) either yield an approximately correct value (near the cusp) or replace one negligible quantity by another. Such a function as $\delta^*(\tau)$ seems

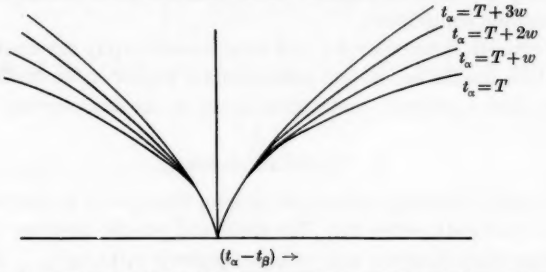

Fig. 1. Sections of a locally homomorphic serial variation surface.

appropriately described as an *acting serial variation function*. Extensive local homomorphy in any finite neighbourhood of $\tau = 0$ is clearly not essential for this representation of (7) in terms of an acting serial variation function; as long as the true function differs from the acting function only by a function which is smooth and changes smoothly under translation the discrepancies will be differenced out, and for this limiting local homomorphy is sufficient.

Smooth heteromorphy of $x(t)$ in the mean $\mu(t)$ implies that $\mu(t)$ changes smoothly with t so that differences such as $[\mu(T+\alpha w)-\mu(T+\alpha w+\theta w)]$ will tend to be small; it would be at once realistic and convenient to assume that the products of these differences which occur in (7) are of the same order of magnitude as the effects of smooth heteromorphy on the differences of serial variation parameters. Provided that the smoothness is sufficient, we may then use the formula (8), which is exact for stationary series, as an approximation to (7), interpreting $\delta^*(\tau)$ as an acting serial variation function. The resulting error in var U is then merely the average of the errors in the separate terms of (8).

An important practical consequence of this and other similar results is that the assumption of stationarity involved in the specification of structure of observed data may be regarded as 'robust', applying Box's concept of robustness (Box (1953)) to this different type of situation. Departures from stationarity will often be in the direction of smooth heteromorphy, and physical conditions often justify this assumption. The serial variation statistics

$$d(\tau) = \text{Av}\, \tfrac{1}{2}[x(t)-x(t+\tau)]^2, \qquad (9)$$

obtained by averaging semi-squared differences, can reasonably be used as a basis for estimating $\delta^*(\tau)$; this function may be fitted to them as for stationary series, since they too have sampling properties which may be shown to be robust under smoothly heteromorphic departures from stationarity; furthermore, if $x(t)$ is locally homomorphic, they will have real descriptive meaning for the smaller values of τ. Tests and measures of heteromorphy would not be difficult to devise, but for the practising statistician they would fill only a rather subsidiary need, like tests of normality in classical statistics; he would usually be content to assume smooth heteromorphy in the absence of conspicuous evidence against it.

Again, the concept of smooth heteromorphy reduces the need to incorporate the somewhat indeterminate notion of *trend* into specifications for data, particularly when such trend would not be evolutionary in character, but merely a smooth slow movement with unspecified variational properties. It is not usually possible to distinguish such a trend from the longer term aspects of serial correlation in observed data; fitting it is apt to be an arbitrary procedure, and in analysis parameters have to be introduced to specify it. Accordingly, it is useful to be able to dispense with it, in this local type of analysis at least; in the specifications required it is usually possible to replace trend plus stationary component by a single smoothly heteromorphic component.

The concepts of smooth heteromorphy and local homomorphy are readily generalized to concomitant variables, spatial series, and parameters of higher order (leading to the concept of local normality); such a general treatment is given in the next section.

3. General theory

For simplicity of notation and exposition, the general theory will be developed only for the case of a univariate stochastic series $x(t)$. The ideas and proofs, however, extend readily to the case of a vector random function $\mathbf{x}(\mathbf{t})$ with components $x_1(\mathbf{t}), x_2(\mathbf{t}), \ldots$ defined at points $\mathbf{t}$ of a multidimensional space S (cf. Jowett (1955*a*)), there being no concept which is not obvious that occurs in passing to a vector function and to a higher dimension.

The means $\mu(t)$, standard deviations $\sigma(t)$, covariances $\mathrm{cov}\,(x(t_\alpha), x(t_\beta))$ and serial variation functions $\delta(t_\alpha, t_\beta)$, are all instances of the general concept of *probability parameter function* $\lambda(t_1, t_2, \ldots)$ defining any specified parameter of the multivariate probability distribution of $x(t_1), x(t_2), \ldots$ as a function of $t_1, t_2, \ldots$; this concept also covers higher order cumulants.

Definition of smooth heteromorphy: The stochastic series $x(t)$ is smoothly heteromorphic in the probability parameter function $\lambda(t_1, t_2, \ldots, t_m)$ of $m(\geqslant 2)$ phase values $t_1, t_2, \ldots, t_m$ in the interval R if, for all $t_1, t_2, \ldots, t_m$ in R, there exist a function $\lambda^*(t_1 - t_2, t_2 - t_3, \ldots)$ of separation only such that $\lambda - \lambda^*$ and its first and second derivatives with respect to $t_1, t_2, \ldots$ are absolutely bounded in R by constants $M_{\lambda 0}, M_{\lambda 1}, M_{\lambda 2}$, respectively. The smoothness of the heteromorphy is characterized by the smallness of $M_{\lambda 1}, M_{\lambda 2}$ which can be achieved by suitable choice of λ^*.

For functions of a single phase value, λ^* will be taken as a constant.

If we write

$$\alpha(t_1, t_2, \ldots) \equiv \lambda(t_1, t_2, \ldots) - \lambda^*(t_1 - t_2, t_2 - t_3, \ldots), \tag{10}$$

this definition implies that if a is a constant having the dimensions of t and such that $|\tau_1|, |\tau_2|, \ldots < a$, and if the interval $[(t_1, t_2, \ldots), (t_1 + \tau_1, t_2 + \tau_2, \ldots)]$ belongs to R, λ admits of a representation in the form

$$\lambda(t_1 + \tau_1, t_2 + \tau_2, \ldots) = \lambda^*(t_1 + \tau_1 - t_2 - \tau_2, \ldots) + \left(1 + \tau_1 \frac{\partial}{\partial t_1} + \tau_2 \frac{\partial}{\partial t_2} + \ldots\right) \alpha(t_1, t_2, \ldots) + \tfrac{1}{2} m^2 a^2 \theta_\lambda(t_1 + \tau_1, t_2 + \tau_2, \ldots)\, M_{\lambda 2}, \tag{11}$$

where $|\theta_\lambda(t_1+\tau_1, t_2+\tau_2, \ldots)| \leqslant 1, \alpha \leqslant M_{\lambda 0}, \partial\alpha/\partial t_1, \ldots \leqslant M_{\lambda 1}$. The function λ^* will be called an *acting parameter function*; it need not itself possess bounded second derivatives everywhere in R.

If $x(t)$ is smoothly heteromorphic in all cumulants of the rth and lower orders, it will be described as smoothly heteromorphic to the rth order. In many applications, smooth heteromorphy to the second order is all that is necessary, and this can easily be shown to imply smooth heteromorphy in the serial variation function $\delta(t_1, t_2)$, leading to the concept of an *acting serial variation function* $\delta^*(\tau)$, which in practice will usually be found to have a cusp at $\tau = 0$.

We shall be concerned with intervals of t of width $2a$ (later to be equated to twice the constant a mentioned above), and with seminvariant linear local functions (s.l.l.f.'s) $L_\alpha(t)$ of diameter $2a$ defined by Stieltjes integrals as follows:

$$L_\alpha(t) \equiv \int_{-a}^{+a} x(t+\tau)\, dl_\alpha(\tau), \tag{12}$$

where

$$\int_{-A}^{+A} dl_\alpha(\tau) = 0 \quad (\text{all } A \geqslant a), \tag{13}$$

justifying the use of the term seminvariant. With any s.l.l.f. is associated its *absolute coefficient sum*

$$H_\alpha = \int_{-a}^{+a} |\, dl_\alpha(\tau)\,| \tag{14}$$

and *first absolute coefficient moment*

$$K_\alpha = \frac{1}{2a}\int_{-a}^{+a} |\,\tau\,|\,|\, dl_\alpha(\tau)\,|. \tag{15}$$

THEOREM 1. *If, for* $[t_1-a, t_2-a, \ldots), (t_1+a, t_2+a, \ldots)] \subset R$, *the heteromorphy of the series* $x(t)$ *in the probability parameter function* $\lambda(t_1, t_2, \ldots, t_m)$ *is sufficiently smooth and if* $L_\alpha(t_1), L_\beta(t_2), \ldots$ *are* m *localized seminvariant linear local functions of sufficiently small maximum diameter* $2a$ *having finite absolute coefficient sums* $H_\alpha, H_\beta, \ldots$, *then*

$$\left|\int_{-a}^{+a}\cdots\int_{-a}^{+a} \lambda(t_1+\tau_1, t_2+\tau_2, \ldots)\, dl_\alpha(\tau_1)\, dl_\beta(\tau_2)\cdots - \int_{-a}^{+a}\cdots\int_{-a}^{+a} \lambda^*(t_1+\tau_1-t_2-\tau_2, \ldots)\, dl_\alpha(\tau_1)\, dl_\beta(\tau_2)\cdots\right| < \epsilon, \tag{16}$$

where ϵ is a preassigned small positive quantity.

Proof. When the right-hand side of (11) is substituted in the left-hand integral in (16), the seminvariant property of the L's ensures that all terms not involving all of $\tau_1, \tau_2, \ldots$ yield integrals which are identically zero.

$$\left|\int_{-a}^{+a}\cdots\int_{-a}^{+a} \tfrac{1}{2}m^2a^2\theta_\lambda(t_1+\tau_1, t_2+\tau_2, \ldots)\, M_{\lambda 2}\, dl_\alpha(\tau_1)\, dl_\beta(\tau_2)\cdots\right| \leqslant \tfrac{1}{2}m^2a^2 M_{\lambda 2} H_\alpha H_\beta \cdots \tag{17}$$

and the quantity on the right-hand side of (17) will be less than ϵ if $M_{\lambda 2}$ and a are small enough. The theorem follows.

In Theorem 1 and subsequent theorems we are not in fact thinking of ϵ in the customary way as a quantity ultimately to be made vanishingly small, but as a quantity which must be acceptably small for the theorems to be applied, its size being governed jointly by the smoothness of the heteromorphy of a given series and the smallness of diameter of the s.l.l.f.'s

involved in the statistical analysis which it is proposed to apply to it. Sometimes, as for example in Jump analysis (Jowett (1955*d*) and Jowett & Wright (in preparation)), it is the diameter of the s.l.l.f.'s which has to be made sufficiently small to make the theorems apply in the case of a given series, and thus make statistical analysis relevant to some issue possible at all.

THEOREM 2. *If, for* $[(t_1-a, t_2-a, \ldots), (t_1+a, t_2+a, \ldots)] \subset R$, *the heteromorphy of the series* $x(t)$ *is sufficiently smooth to the second order, the sampling covariance of any pair of s.l.l.f.'s* $L_\alpha(t_1)$, $L_\beta(t_2)$ *localized at points* t_1, t_2 *of* R *and having finite absolute coefficient sums* H_α, H_β *and first moments* K_α, K_β *may be made to satisfy an inequality of the form*

$$\left|\operatorname{cov}[L_\alpha(t_1), L_\beta(t_2)] - \int_{-a}^{+a}\int_{-a}^{+a} -\delta^*(t_1+\tau_1-t_2-\tau_2)\,dl_\alpha(\tau_1)\,dl_\beta(\tau_2)\right| < \epsilon, \tag{18}$$

where $\delta^*(\tau)$ *is an acting serial variation function of separation only,* a *is the greater of the semi-diameters of the two s.l.l.f.'s, and* ϵ *is a preassigned small positive quantity.*

Proof.

$$\operatorname{cov}[L_\alpha(t_1), L_\beta(t_2)] = \int_{-a}^{a}\int_{-a}^{a} \operatorname{cov}[x(t_1+\tau_1), x(t_2+\tau_2)]\,dl_\alpha(\tau_1)\,dl_\beta(\tau_2) \tag{19}$$

$$= \int_{-a}^{a}\int_{-a}^{a} \{\tfrac{1}{2}\sigma^2(t_1+\tau_1) + \tfrac{1}{2}\sigma^2(t_2+\tau_2) - \delta(t_1+\tau_1, t_2+\tau_2) + \tfrac{1}{2}[\mu(t_1+\tau_1) - \mu(t_2+\tau_2)]^2\}\,dl_\alpha(\tau_1)\,dl_\beta(\tau_2), \tag{20}$$

$$= \int_{-a}^{a}\int_{-a}^{a} \{-\delta(t_1+\tau_1, t_2+\tau_2) + \tfrac{1}{2}[\mu(t_1+\tau_1) - \mu(t_2+\tau_2)]^2\}\,dl_\alpha(\tau_1)\,dl_\beta(\tau_2), \tag{21}$$

the integrals of the terms involving τ_1 or τ_2 only vanishing because of the seminvariant property of the s.l.l.f.'s.

In (21) we may expand the square. Because of the seminvariant property, the integrals of the terms in $\mu^2(t_1+\tau_1)$ and $\mu^2(t_2+\tau_2)$ vanish. Hence, using (11) and the seminvariant property,

$$\begin{aligned}\operatorname{cov}[L_\alpha(t_1), L_\beta(t_2)] = \int_{-a}^{+a}\int_{-a}^{+a} &\{-\delta^*(t_1+\tau_1-t_2-\tau_2) - 2a^2 M_{\delta 2}\theta_\delta(t_1+\tau_1, t_2+\tau_2)\}\,dl_\alpha(\tau_1)\,dl_\beta(\tau_2) \\ &- \int_{-a}^{+a}\left\{\tau_1\frac{\partial\mu(t_1+\tau_1)}{\partial t_1} + \tfrac{1}{2}a^2 M_{\mu 2}\theta_\mu(t_1+\tau_1)\right\}dl_\alpha(\tau_1) \\ &\times \int_{-a}^{+a}\left\{\tau_2\frac{\partial\mu(t_2+\tau_2)}{\partial t_2} + \tfrac{1}{2}a^2 M_{\mu 2}\theta_\mu(t_2+\tau_2)\right\}dl_\beta(\tau_2).\end{aligned} \tag{22}$$

It follows that

$$\begin{aligned}&\left|\operatorname{cov}[L_\alpha(t_1), L_\beta(t_2)] - \int_{-a}^{+a}\int_{-a}^{+a} -\delta^*(t_1+\tau_1-t_2-\tau_2)\,dl_\alpha(\tau_1)\,dl_\beta(\tau_2)\right| \\ &\qquad \leqslant 2a^2 M_{\delta 2} H_\alpha H_\beta + (2aK_\alpha + \tfrac{1}{2}a^2 M_{\mu 2} H_\alpha)(2aK_\beta + \tfrac{1}{2}a^2 M_{\mu 2} H_\beta).\end{aligned} \tag{23}$$

If $M_{\delta 2}$, $M_{\mu 2}$ and a are sufficiently small, the right-hand side of (23) will be less than ϵ. This proves the theorem.

Definition of acting normality. The stochastic series $x(t)$ *possesses the property of acting normality in the interval* R *if it is smoothly heteromorphic in* R *with acting cumulants of order higher than the second which may be taken as zero.*

THEOREM 3. *If in an interval R the heteromorphy of $x(t)$ is sufficiently smooth to order s, and if $x(t)$ has the property of acting normality, then for any set of s s.l.l.f.'s with bounded absolute coefficient sums and first moments, with diameters not greater than $2a$, and involving only phase values lying in R,*

$$E[L_{\alpha_1}(t_1)\, L_{\alpha_2}(t_2) \dots L_{\alpha_s}(t_s)] = \begin{cases} \Sigma C_{1p} C_{qr} \dots C_{ms} + O(\epsilon) & (s \text{ even}) \\ O(\epsilon) & (s \text{ odd}) \end{cases}, \tag{24}$$

where

$$C_{qr} = \int_{-a}^{+a}\int_{-a}^{+a} -\delta^*(t_q+\tau_q-t_r-\tau_r)\, dl_{\alpha_q}(\tau_q)\, dl_{\alpha_r}(\tau_r), \tag{25}$$

and the summation is taken over all partitions of the suffixes $1, 2, \dots, s$ into different pairs.

Proof. Let $\kappa(t_u t_v \dots t_z)$ be the cumulant of highest order associated with the product $x(t_u)\, x(t_v) \dots x(t_z)$, repetition of arguments being permissible. Thus, for example

$$\kappa(t, t+\tau) \equiv \operatorname{cov}(x(t), x(t+\tau)),\ \kappa(t,t) \equiv \sigma^2(t).$$

From the relationship between the moments and cumulants of a multivariate distribution, it may easily be shown that

$$E(x(t_u)\, x(t_v) \dots x(t_z)) = \Sigma\kappa(t_u, \dots)\, \kappa(\dots) \dots \kappa(\dots), \tag{26}$$

where the summation is of all products of cumulants obtainable as follows: the symbols are partitioned into one or more subsets; the symbols of each subset are taken as the arguments of a cumulant; and all the cumulants resulting from the particular partitioning are multiplied together. Thus, for example,

$$E(x(t_u)\, x(t_v)\, x(t_w)) = \kappa(t_u, t_v, t_w) + \kappa(t_u)\, \kappa(t_v, t_w) + \kappa(t_u, t_v)\, \kappa(t_w) + \kappa(t_u, t_w)\, \kappa(t_v) + \kappa(t_u)\, \kappa(t_v)\, \kappa(t_w). \tag{27}$$

The constitution of the summation is unaffected by coincidences in the actual values $t_u, t_v, \dots, t_z$. Now

$$E[L_{\alpha_1}(t_1)\, L_{\alpha_2}(t_2) \dots L_{\alpha_s}(t_s)]$$

$$= \int_{-a}^{+a} \dots \int_{-a}^{+a} E[x(t_1+\tau_1)\, x(t_2+\tau_2) \dots x(t_s+\tau_s)]\, dl_{\alpha_1}(\tau_1) \dots dl_{\alpha_s}(\tau_s), \tag{28}$$

$$= \int_{-a}^{+a} \dots \int_{-a}^{+a} [\Sigma\kappa(t_1+\tau_1, \dots)\, \kappa(t_p+\tau_p, \dots) \dots]\, dl_{\alpha_1}(\tau_1) \dots dl_{\alpha_s}(\tau_s), \tag{29}$$

$$= \Sigma\left[\int_{-a}^{+a} \dots \int_{-a}^{+a} \kappa(t_1+\tau_1, \dots)\, dl_{\alpha_1}(\tau_1) \dots\right]\left[\int_{-a}^{+a} \dots \int_{-a}^{+a} \kappa(t_p+\tau_p, \dots)\, dl_{\alpha_p}(\tau_p) \dots\right], \tag{30}$$

the summations in (28), (29), (30) being defined as in (26). If the heteromorphy of $x(t)$ is sufficiently smooth, and a sufficiently small, every integral in (30) may be replaced, to the order of a given ϵ, by that of an acting cumulant which is a function of separation only. Furthermore, if $x(t)$ has acting normality, only the acting cumulants of the second order survive. Hence, if we neglect terms of order ϵ, the only terms of (34) which make a contribution are those consisting entirely of products of integrals of second order cumulants. Now typically

$$\int_{-a}^{a}\int_{-a}^{a} \kappa(t_q+\tau_q, t_r+\tau_r)\, d\tau_q d\tau_r = \operatorname{cov} L_{\alpha_q}(t_q), L_{\alpha_r}(t_r) \tag{31}$$

$$= \int_{-a}^{a}\int_{-a}^{a} -\delta^*(t_q+\tau_q-t_r-\tau_r)\, dl_{\alpha_q}(\tau_q)\, dl_{\alpha_r}(\tau_r) + 0(\epsilon) \tag{32}$$

by Theorem 2. The proof of the theorem then follows at once. Formula (32) differs from the corresponding formula for stationary series, established in (1955*a*), only in the presence of the term $O(\epsilon)$ and the use of an acting, instead of actual, serial variation function of separation only. Accordingly, if $\delta^*(\tau)$ has sufficient tendency towards linearity with increasing $|\tau|$, the methods of that paper may be applied and lead to the following theorem, which is a suitably modified form of the theorem established in § 2 of that paper, and which is proved in the same way:

THEOREM 4. *The expectation of any product of powers of s.l.l.f.'s of maximum diameter* $2a$ *of values in an interval* R *of a stochastic series* $x(t)$ *having, in* R,

(*a*) *heteromorphy which is sufficiently smooth,*

(*b*) *acting normality,*

(*c*) *an acting serial variation function which has a sufficiently rapid tendency to linearity as the separation* $|\tau|$ *increases beyond* a,

may be expressed as a sum of terms which are either of magnitude at most of specified order ϵ, *or involve only values of the acting serial variation function in a neighbourhood of* $|\tau| = 0$ *having a magnitude of order* a, *i.e. depending on local variational properties of* $x(t)$.

Many statistics which are of practical interest are means, or functions of means of products of powers of n s.l.l.f.'s which are evenly spread over a region of S and their asymptotic sampling moments depend ultimately on the n^2 covariances of the s.l.l.f.'s involved in them. Because of the even spread, the number of these which are not *far apart* (a term defined precisely in (1955*a*)) is of magnitude $O(n)$. If $x(t)$ has the properties specified in Theorem 4 it is only the covariances of these which are not negligible, and these may be expressed in terms of what are essentially local variational properties, as indicated in Theorem 4. Following the arguments of (1955*a*) we may assert the principle that *for smoothly heteromorphic series the sampling properties of local statistics depend essentially on local properties; in general the same sampling formulae may be used as in stationary series, with the acting serial variation function playing the role of the serial variation function in stationary series.*

It will be observed that none of the theorems explicitly involve the concept of *local homomorphy*, which was mentioned in § 2; it is not necessary for their application that λ^* should approximate to λ in any sense. Nevertheless, local homomorphy is a useful property for a series to have; it is consistent with smooth heteromorphy, and implies the possibility of such an approximation, namely that λ^* may be chosen to approximate to λ when the separations of the arguments $(t_1, t_2, \ldots)$ involved are of order not greater than a. Accordingly it is worthy of formal definition, which is conveniently given in terms of the corresponding limiting property.

Definition of limiting local homomorphy. The stochastic series $x(t)$ *is locally homomorphic in the limit in the interval* R *if for any* $\epsilon > 0$ *there exists a magnitude* a, *depending on* ϵ, *such that whenever* $[t_1, t_2, \ldots, t_m]$ *lies in* R *and can be included in some interval of width* $2a$, *an acting probability parameter function* $\lambda^*(t_1 - t_2, \ldots)$ *can be found which is such that*

$$|\lambda(t_1, t_2, \ldots) - \lambda^*(t_1 - t_2, \ldots)| < \epsilon. \tag{33}$$

Limiting local homomorphy will be taken to imply some degree of local homomorphy, and carries with it the possibility of actually approximating to λ by means of λ^*; the extent of the local homomorphy is characterized by the nature of the approach to the limit, a large value of a for a given ϵ implying greater local homomorphy than a small one.

For some probability parameter functions, smooth heteromorphy is sufficient for limiting local homomorphy. An important instance is the serial variation function $\delta(t_1, t_2)$. The definition of smooth heteromorphy for $[t_1, t_2]\ \epsilon\, R$ permits us to choose δ^* so that $\delta^*(0) = 0$. Writing

$$\phi(t_1, t_2) \equiv \delta(t_1, t_2) - \delta^*(t_1 - t_2), \tag{34}$$

from the boundedness of the second derivatives of ϕ it follows that for some t_1', t_2' lying in the interval t_1, t_2

$$\phi(t_1, t_2) = (t_1 - \bar{t})\frac{\partial \phi}{\partial t_1}(t_1', t_2) + (t_2 - \bar{t})\frac{\partial \phi}{\partial t_2}(t_1, t_2'), \tag{35}$$

where $\bar{t} = \frac{1}{2}(t_1 + t_2)$. If $|t_1 - t_2| < 2a$, where $a = \epsilon / 2M_{\delta 1}$,

$$|\phi(t_1, t_2)| < 2aM_{\delta 1} = \epsilon, \tag{36}$$

thus establishing limiting local homomorphy in δ. On the other hand, consideration of the case

$$\lambda(t_1, t_2, t_3) \equiv \delta(t_1, t_2)\,\mu(t_3) \tag{37}$$

shows that smooth heteromorphy is not always sufficient to secure limiting local homomorphy.

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DEPENDENCE OF THE FIDUCIAL ARGUMENT ON THE SAMPLING RULE†

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Suppose that we have a series of observations, given to have been drawn independently from a common chance distribution of specified form depending on a single unknown parameter θ. Suppose the observations have been taken according to some sampling rule such that the eventual number of observations does not depend on θ except possibly through the observations themselves. All recognized kinds of sequential rule, including fixed-sample-size rules, satisfy this condition. (A type of sampling rule to be excluded would be one where the number of observations depended on other observations subsequently suppressed.)

It is well known that the likelihood function of θ, given the observations, is independent of the sampling rule. For in any expression for the chance of observing what has actually been observed, the sampling rule only enters as a factor independent of θ, which is therefore irrelevant when the chances corresponding to different values of θ are compared. It follows that the posterior probability distribution for θ derived by Bayes's theorem from some given prior distribution is also independent of the sampling rule. (I shall refer to this use of Bayes's theorem as 'the Bayesian argument', for short.)

On the other hand, in order to make a significance test (for instance, a test of whether the specified form of parent distribution is reasonably compatible with the observations), we need to take the sampling rule into account, in general. The same is true of other common statistical procedures; for examples and references, see Anscombe (1954).

The purpose of this note is to show that in order to apply Fisher's fiducial argument (as recently re-expounded by him, 1956) we need to consider the sampling rule, just as for significance tests. This is interesting for two reasons.

First, Fisher has stressed that a fiducial distribution is to be interpreted in exactly the same way as if it had been derived by a Bayesian argument, and indeed a fiducial distribution may be taken as prior distribution for incorporation in a subsequent Bayesian argument. Moreover, when the fiducial argument is for mathematical reasons not available, Fisher suggests that the likelihood function may be considered directly instead. Thus, what the fiducial argument is likened to and intimately associated with, and what is suggested as a substitute for it when it is not available, both differ from it in regard to dependence on the sampling rule.

Secondly, Lindley (1957) has shown that, if the common chance distribution of the observations referred to in the opening sentence above is a member of the exponential family, then the fiducial argument does not have a property of consistency evidently desired for it by Fisher unless the resulting distribution for θ is actually a posterior Bayes distribution. The examples below show that, in so far as fiducial distributions can be identified with posterior Bayes distributions, the corresponding prior distributions *depend on the*

† Prepared in connexion with research sponsored by the United States Office of Naval Research.

sampling rule. It is also pointed out that the same applies (in a more striking degree) to decision procedures chosen by the minimax rule, when these are expressed in Bayesian terms.

Thus, Fisher's hierarchy of inferential methods, namely (in ascending order of informativeness), (i) significance tests, (ii) contemplation of the likelihood function, (iii) the fiducial argument, shows a curious oscillation between heeding and ignoring the sampling rule. Items (i) and (iii) require for their definition that the whole sample space be defined; and this property is shared with Neyman's confidence intervals, the sampling distribution of a statistic, and Wald's minimax decision procedures. But for item (ii), as for the classic use of Bayes's theorem, we require to know no more of sample space than the single observed point. It seems to me that, before Fisher's account of statistical reasoning can be accepted as correct, this oscillation must be explained and made acceptable. More generally, anyone who claims correct understanding of statistical inference needs to explain when, why, and how, knowledge of the sampling rule is relevant to the interpretation of given observations.

Examples showing dependence of the fiducial argument on the sampling rule. In constructing such examples, the difficulty is encountered that when the sample size is a random variable its distribution, being discrete, cannot be made the basis for an exact fiducial argument, though for large sample sizes an approximate fiducial argument may be possible. In order to avoid this kind of imprecision, the examples below refer, not to ordinary sampling with a finite number of observations, but to the sampling of a stochastic process with continuous time parameter. Large samples from (respectively) binomial and normal populations will exhibit approximately the same phenomena.

Example 1. Let $r_\theta(t)$ denote a Poisson process with continuous time parameter t and independent increments, such that

$$r_\theta(t+\tau) - r_\theta(t)$$

has a Poisson distribution with mean $\theta\tau$, for any positive τ and any t. Thus the jumps in $r_\theta(t)$ occur at mean rate θ per unit time. We suppose θ to be positive but otherwise unknown. Starting with $r_\theta(0) = 0$, let a realization of the process be observed continuously for a period, at the end of which t has the value T and $r_\theta(t)$ has the value R, say. The likelihood function of θ, given the observations, depends only on the end-point (T, R), being†

$$\theta^R e^{-\theta T}. \tag{1}$$

Suppose we are told that the duration of observation, T, was fixed in advance. Then R is the observed value of a chance variable (for fixed θ) having the Poisson distribution with mean θT. No exact fiducial distribution for θ, given R, can be found, but if R is large we can use an approximate fiducial argument (Fisher, 1956, pp. 62–3), asymptotically equivalent to the use of Bayes's theorem with likelihood (1) and prior distribution‡

$$\frac{d\theta}{\sqrt{\theta}}. \tag{2}$$

Suppose, alternatively, we are told that sampling was 'inverse', R was fixed in advance, and observation stopped at the first instant when the condition $r_\theta(t) = R$ was satisfied.

† The likelihood function is a set of odds for θ, and is therefore arbitrary up to a multiplying factor independent of θ.

‡ Prior distributions need not be normalized, since constant factors cancel out when Bayes's theorem is used.

Then T is the observed value of a chance variable (for fixed θ) having the distribution

$$\frac{\theta^R}{(R-1)!}\, T^{R-1} e^{-\theta T}\, dT. \tag{3}$$

This inverts (see Fisher, 1956, p. 53) to give a fiducial distribution for θ, given T, namely

$$\frac{T^R}{(R-1)!}\, \theta^{R-1} e^{-\theta T}\, d\theta, \tag{4}$$

which is the same as the posterior Bayes distribution for θ derived from (1) with the prior distribution

$$\frac{d\theta}{\theta}. \tag{5}$$

Example 2. This is similar to Example 1 except that now both interpretations of the observations lead to proper fiducial distributions. Let $x_\theta(t)$ denote a normal (Wiener) process with continuous time parameter t and independent increments, such that

$$x_\theta(t+\tau) - x_\theta(t)$$

is normally distributed with mean $\theta\tau$ and variance τ, for any positive τ and any t. Thus $x_\theta(t)$ increases at the rate θ per unit time in mean and 1 per unit time in variance. We suppose θ to be unknown. Starting with $x_\theta(0) = 0$, let a realization of the process be observed continuously for a period, at the end of which t has the value T and $x_\theta(t)$ has the value X, say. We shall suppose that X is positive. The likelihood function of θ, given the observations, depends only on the end-point (T, X), being

$$e^{\theta X - \frac{1}{2}\theta^2 T}. \tag{6}$$

Suppose we are told that the duration of observation, T, was fixed in advance. Then X is the observed value of a chance variable (for fixed θ) normally distributed with mean θT and variance T. This inverts to give a fiducial distribution for θ, given X, namely the normal distribution

$$\sqrt{\left(\frac{T}{2\pi}\right)}\, e^{-\frac{1}{2}T(\theta - X/T)^2}\, d\theta, \tag{7}$$

which is the same as the posterior Bayes distribution for θ derived from (6) with the uniform prior distribution

$$d\theta. \tag{8}$$

Suppose, alternatively, we are told that X was fixed in advance, and observation stopped at the first instant when the condition $x_\theta(t) = X$ was satisfied. Now this specification of the sampling rule is not adequate as it stands, unless there is prior knowledge that $\theta \geqslant 0$; for if $\theta < 0$ the chance that the positive value X will ever be attained by the process is less than one (in fact $e^{2\theta X}$). In the absence of prior knowledge that $\theta \geqslant 0$, the sampling rule may be truncated, thus: observation stops at the first instant when either $x_\theta(t) = c$ or $t = k$, where c and k are given positive numbers. If (T, X) denotes the end-point of the sample path, either $X = c$ or $T = k$. Let

$$T^* = T + c - X,$$

so that $T^* = T$ if $X = c$ and $T^* = k + c - X$ if $T = k$. Then clearly T^* is a sufficient statistic for θ. It is shown below (next paragraph) that the chance distribution for T^*, given θ, has the monotonicity property needed for the derivation of a fiducial distribution for θ, given T^* (Fisher, 1956, p. 69). Note that for any sample path such that $T < k$, the value of k is not needed in the fiducial argument, which is based on the cumulative distribution function of T^*, given θ, at the observed value of T.

To verify the monotonicity property of the distribution of T^*, let $F_\theta(x)$ denote the chance that $T^* \leqslant x$. Then $F_\theta(x)$ is the chance-measure of all sample paths in the interval $0 \leqslant t \leqslant k$ for which $T^* \leqslant x$. Let each of these paths be deformed from $x_\theta(t)$ to $x_\theta(t) + \alpha t$, where α is a positive constant. All the deformed paths satisfy $T^* \leqslant x$, and if θ is changed to $\theta + \alpha$ their measure is still $F_\theta(x)$. In general there are also other paths, besides these deformed paths, for which $T^* \leqslant x$. It follows that $F_{\theta+\alpha}(x) \geqslant F_\theta(x)$, i.e. $F_\theta(x)$ is an increasing function of θ as well as of x. For given positive x, $F_\theta(x)$ runs from 0 to 1 as θ runs from $-\infty$ to $+\infty$; for given θ, it runs from 0 to 1 as x runs from 0 to ∞.

Suppose, then, we are told that the given sample path was observed under a sampling rule of the sort just described, and that $x_\theta(t)$ attained its prescribed upper bound (therefore equal to X) before the truncation provision came into effect. The chance distribution for T, given θ and X, is (see Feller, 1950, p. 296)

$$\frac{X}{\sqrt{(2\pi)}} T^{-\frac{3}{2}} e^{-\frac{1}{2}T(\theta - X/T)^2} dT. \tag{9}$$

(The integral of this from 0 to the truncation time k is the chance that the process attains the bound X before truncation.) The fiducial distribution for θ, given T, derived from (9) is not a posterior Bayes distribution for any prior distribution, because the condition given by Lindley is not satisfied (namely, that there exist transformations of T to U and of θ to ϕ, such that ϕ is a location parameter for U). However, we shall see that if X is large the fiducial distribution is approximately a posterior Bayes distribution.†

If we integrate (9) with respect to T and then take the differential with respect to θ, we obtain the fiducial distribution for θ in the form

$$d\theta \int_0^T \frac{X}{\sqrt{(2\pi)}} t^{-\frac{3}{2}} (X - \theta t) e^{-\frac{1}{2}t(\theta - X/t)^2} dt.$$

After changing the variable of integration from t to $\sqrt{t}(\theta + X/t)$, we obtain the explicit expression

$$2X e^{2X\theta} \Phi(-V) d\theta, \tag{10}$$

where $$V = \sqrt{T}(\theta + X/T) \quad \text{and} \quad \Phi(t) = \int_{-\infty}^{t} e^{-\frac{1}{2}t^2} dt / \sqrt{(2\pi)}.$$

We now restrict attention to values of θ near X/T, according to the asymptotic specification: $X \to \infty$, $T \to \infty$, with X/T constant, $\sqrt{T}\,|\,\theta - X/T\,|$ bounded. On using the asymptotic expansion for $\Phi(-V)$ as $V \to \infty$, we see that (10) is

$$\frac{1}{\sqrt{(2\pi)}} \frac{2X}{V} e^{-\frac{1}{2}T(\theta - X/T)^2} \left\{1 + O\left(\frac{1}{T}\right)\right\} d\theta.$$

It is easy to show that

$$\frac{V^2}{4X} = \theta + O\left(\frac{1}{T}\right),$$

and so finally we have the fiducial distribution for θ in the form

$$\sqrt{\left(\frac{X}{2\pi\theta}\right)} e^{-\frac{1}{2}T(\theta - X/T)^2} \left\{1 + O\left(\frac{1}{T}\right)\right\} d\theta. \tag{11}$$

† The following proof, shorter than my original one, is due to Dr H. E. Daniels.

This is asymptotically the posterior Bayes distribution for θ derived from (6) with the prior distribution

$$\frac{d\theta}{\sqrt{|\theta|}}. \tag{12}$$

(This has been established for values of θ near X/T. The insertion of the modulus sign in the denominator extends the result to include what we should have obtained if we had supposed X to be negative.)

Remarks. (1) In Example 1, if we had not been given the complete observational record, but only the end-point (T, R), to be told that the sampling rule was the inverse one would give us an extra piece of information about the sample path, namely, that the ordinate jumped from $R-1$ to R at the final instant T. This extra information is reflected in the fiducial argument (when we translate it into Bayesian terms) by the substitution of the prior distribution (2) by (5), which has the effect of attaching more weight to lower values of θ. Thus, the fiducial argument does not use the observations only in the form of the likelihood function, the 'sufficient' statistics (T, R) are not indeed sufficient for it; further information about the observations, not contained in the likelihood function, is required.† This, of course, is in contrast with the classic use of Bayes's theorem, where the prior distribution, however it may be determined, is wholly independent of the observations used to calculate the likelihood function. (The same remark, with obvious changes, applies equally to Example 2.)

This point applies to other kinds of statistical argument besides the fiducial; for instance, to minimax decision procedures. From the mere fact that a procedure *can* be expressed in terms of Bayes's theorem with some specially-chosen prior distribution, we cannot infer that the procedure ignores the sampling rule, because the 'prior' distribution may depend on the sampling rule (and so not be prior at all). This is illustrated by the minimax procedure for estimating a binomial chance θ from a sample of fixed size n, when the loss in quoting the estimate $\hat{\theta}$ is proportional to $(\hat{\theta}-\theta)^2$. The minimax $\hat{\theta}$ is what would be obtained by the Bayesian argument if the prior distribution for θ were

$$[\theta(1-\theta)]^{\frac{1}{2}\sqrt{n}-1}\,d\theta,$$

which depends on n. (See Hodges & Lehmann, 1950.)

(2) Another way of regarding the examples is as problems in recognizing ancillary statistics. In Example 1, should we think of T as ancillary to R, or of R as ancillary to T?

(3) It is of some interest to inquire how near the above examples come to being unique of their kind. In both cases, other sequential sampling rules besides the 'inverse' one could have been considered (mathematical difficulty permitting), leading to yet other fiducial distributions. Apart from that, the examples do seem to be unique, in the sense that the Poisson and normal processes are (I believe) the only processes with continuous time parameter and stationary independent increments from which examples of similar simplicity could be constructed. Examples 1 and 2 both have the properties: (i) the end-point of the sample path is 'sufficient' for the unknown parameter θ, (ii) there are sequential sampling rules defined by a boundary such that (with chance equal to 1) the sample path attains the boundary exactly without jumping over it, and the time at which the boundary is attained is a 'sufficient' statistic for θ. Both these conditions are needed for simple application of

† A statistic that determines the likelihood function has been termed 'sufficient' by Fisher. The definition carries the innuendo that the likelihood function is all we need to know.

the fiducial argument. Condition (ii) is not satisfied unless the process is either continuous (and therefore normal) or purely discrete, all jumps having the same magnitude (as with the Poisson process). A process of this latter sort, having negative as well as positive jumps, is the difference of two independent Poisson processes. There are now two parameters. However we select one parameter to be estimated, the process cannot (I think) be made to satisfy (i).

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TESTS FOR RANK CORRELATION COEFFICIENTS. I

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1. Purpose of the study

(1·1) *The measures considered*

The following is a first report on an investigation which became possible with the availability of the 25,000 sets of correlated random normal deviates, 3000 of which were published in Fieller, Lewis & Pearson's (1955) *Tracts for Computers*, no. XXVI. The object which we set ourselves was to study with the aid of these data the sampling distributions of, and relationships between, three measures of rank correlation, in the case where the basic variables which have been ranked follow bivariate normal distributions.

We shall use the following notation. Suppose that there are n pairs of associated rankings

$$u_1, u_2, \ldots, u_n \quad \text{and} \quad v_1, v_2, \ldots, v_n,$$

where the integers u_i $(i = 1, 2, \ldots, n)$ may be taken in ascending order $1, 2, \ldots, n$ and the v_i are a permutation of these integers. We shall consider in the present paper the two following measures of correlation between these rankings:

(*a*) *Spearman's coefficient* which we denote by r_S. This is simply the product moment correlation coefficient of u_i, v_i and may be computed from the sum of squared differences

$$S_S = \sum_{i=1}^{n} (u_i - v_i)^2, \tag{1}$$

where

$$r_S = 1 - 6S_S/(n^3 - n). \tag{2}$$

(*b*) *Kendall's coefficient*, τ, which we denote by r_K. This may be computed as follows. For every integer u_i count the number of v_j with $v_j > u_i$ and $j > i$; then add these counts to obtain the positive score P_K. Then

$$r_K = 4P_K/(n^2 - n) - 1. \tag{3}$$

Both r_S and r_K lie between $+1$ and -1. We shall not be concerned here with ties among the u's or v's.

The following is a third coefficient which has been computed for all the sampling data and which we hope to consider later:

(*c*) *The Fisher–Yates coefficient.* Let $\xi(i \mid n)$ be a so-called normal order statistic, i.e. the expected value of the ith largest standardized deviate in a sample of n observations from a normal population. Then we may attach these score values to both the u rankings and the v rankings. Fisher & Yates (1938, p. 50) have suggested that a measure of rank correlation might be obtained from the product moment correlation coefficient of these scores, namely

$$r_F = \sum_{i=1}^{n} \xi(i \mid n)\, \xi(v_i \mid n) \Big/ \sum_{i=1}^{n} \xi^2(i \mid n). \tag{4}$$

Convenient tables of the individual $\xi(i \mid n)$ as well as of $\sum_i \xi^2(i \mid n)$ are given, for example, in Fisher & Yates (1938, Tables XX and XXI). As an approximation to the actual product-moment correlation coefficient, r_{xy}, in a normal sample r_F clearly has much to recommend it; but the only discussions of this coefficient of which we are aware are those by Jeffreys (1948, pp. 209–10) and Hoeffding (1951, pp. 86–9).

(1·2) *Some known results on the distribution theory of r_S and r_K*

For a comprehensive summary of the older results, the reader may consult Kendall (1948) and Moran (1950). Briefly these are as follows:

For independent random rankings (i.e. for random permutations of the v_i) the complete distributions of r_S and r_K have been obtained for small n by combinatorial enumeration. Adequate approximations have been evolved for larger n.

In the case of correlated rankings it is first necessary to specify the nature of the dependence. A discussion of this problem of appropriate population models was given by Daniels (1950), and very recently Mallows (1957) has developed a new form of approach related to paired-comparison theory. In the present paper we start from the assumption that the n pairs of rankings u_i, v_i have arisen as the rank numbers in a sample of n pairs of correlated normal variates. Thus, if x_i, y_i $(i = 1, 2, \ldots, n)$ denote a random sample of n paired observations from a bivariate normal population having correlation coefficient ρ, we suppose that the x_i are arranged in order of magnitude and that v_i is the rank of y_i. This model has received considerable attention and a certain number of theoretical results are known. Thus we have

$$\mathscr{E}(r_S) = \frac{6}{(n+1)\pi}\{\sin^{-1}\rho + (n-2)\sin^{-1}\tfrac{1}{2}\rho\} \quad \text{(Moran, 1948)}, \tag{5}$$

$$\operatorname{var}(r_S) = 1/n\{1 - 1{\cdot}563465\rho^2 + 0{\cdot}304743\rho^4 + 0{\cdot}155286\rho^6 + 0{\cdot}061552\rho^8 + 0{\cdot}022099\rho^{10} + \ldots\}. \tag{6}$$

Equation (6) is a large sample approximation due to Kendall (1949) and David, Kendall & Stuart (1951). As we shall see below, it does not appear to be very accurate when the sample size is as small as 10. Turning to r_K, we have

$$\mathscr{E}(r_K) = \frac{2}{\pi}\sin^{-1}\rho \quad \text{(Greiner, 1909)}, \tag{7}$$

$$\operatorname{var}(r_K) = \frac{2}{n(n-1)}\left[1 - \left(\frac{2}{\pi}\sin^{-1}\rho\right)^2 + 2(n-2)\left\{\frac{1}{9} - \left(\frac{2}{\pi}\sin^{-1}\tfrac{1}{2}\rho\right)^2\right\}\right] \quad \text{(Esscher, 1924)}. \tag{8}$$

As far as we are aware, no results are available for the higher moments or cumulants of r_S or r_K, but Sundrum (1953) showed how the third and fourth moments of r_K might be obtained in the general case. He also used some random sampling results to give empirical values for these moments, assuming underlying normal correlation, in the single case $\rho = 1/\sqrt{2}$.

As can be seen from equations (6) and (8), the standard deviations of r_S and r_K change with ρ. Further, as might be anticipated from the parallel case of the product moment correlation coefficient r_{xy}, the shape of the sampling distributions are found to change with ρ. Thus, when we get away from the problem of using rank correlation coefficients in tests of independence, we at once run into difficulties. The lack of results for dependent rankings has made it difficult to compare the relative merits of different rank coefficients in detecting dependence, nor has it been possible to use these coefficients for a comparison of correlation in different populations. If we accept the underlying bivariate normal struc-

ture, then we are faced with the distributional problem; if we do not accept this, then we have also to look for a simple definition of non-parametric dependence.

(1·3) *The present results and their bearing on these difficulties*

While we do not claim to have solved all these difficulties we hope, in this paper, to have compiled evidence which shows that the problem is capable of a simple solution provided the rankings arise from the class of population models specified below. We proceed as follows:

(A) We start with rankings generated by sampling from a bivariate normal parent with correlation ρ. With the help of extensive sampling experiments backed by analytical approximation, we show that if n is not too large the z-transforms

$$z_S = \tanh^{-1} r_S = \tfrac{1}{2} \log_e \frac{1+r_S}{1-r_S}, \quad z_K = \tanh^{-1} r_K \tag{9}$$

are approximately normally distributed with variances nearly independent of ρ. In fact

$$\text{var}\,(z_S) \sim \frac{1{\cdot}060}{n-3}, \quad \text{var}\,(z_K) \sim \frac{0{\cdot}437}{n-4}. \tag{10}$$

The expectation of z_K can be expressed approximately as a simple function of ρ, making use of the expressions for $\mathscr{E}(r_K)$ and var (r_K) given in (7) and (8). The approximation to the expectation of z_S is less satisfactory in small samples owing to the inadequacy of the expression (6) for var (r_S)*. It should be noted, however, that, just as in using the z-transformation for r_{xy}, a knowledge of the precise expectation of the transformed variable is not necessary in a number of the test procedures that become available.

(B) The results in A can clearly be extended to a much wider class of parental distributions. If we start from a bivariate normal distribution of x, y and introduce new variates $X = f(x)$, $Y = g(y)$, the rankings of X and Y will clearly be identical with those of x and y provided the functions f and g are monotonic. Thus, the simple results under A will also apply to rankings generated by the wider class of bivariate distributions of X, Y. Conversely, starting from any bivariate distribution $\phi(X, Y)$ we can always find monotonic transformations $X = f(x)$, $Y = g(y)$ to standardized normal variates x and y. The resulting bivariate distribution $\psi(x, y)$ will not necessarily be bivariate normal, but we think it likely that in practical situations it would not differ greatly from this form.† This is a field in which further investigation would be of considerable interest.

(C) Summarizing the results of A and B, we may state that if the rankings are generated by one of a wide class of distributions of paired variables‡ X, Y, then the z transforms of the rank correlation measures z_S, z_K can be regarded as normal variates with variances dependent only on the sample size, and given approximately in equations (10). Further, within this class of bivariate populations, either of the z transforms is an unbiased estimate of a function of the correlation ρ. This is the correlation in the bivariate distribution obtained after distortion to normality of the marginal distribution of X and Y. ρ may be regarded as a non-parametric measure of dependence. Without the need to specify ρ, simple tests of

* The approximation to the expectation of z contains the variance of r; see equation (13) below.

† Johnson (1949) considered a particular case of surfaces having the property of being convertible into the bivariate normal form through the application of his S_B and S_U transformations to the marginal distributions.

‡ It is of course realized that other models of non-parametric dependence have been suggested in which the ranks are not generated by a parental bivariate distribution. Such models are not considered here.

significance may be applied to the z values to determine whether two or more samples are likely to have come from populations with a common ρ.

(D) Within these conditions it is possible to make approximate comparisons of the relative merits of the rank coefficients r_S and r_K (and later we hope of r_F). In particular, we may compare their power in detecting differences in population ρ values.

2. THE EXPERIMENTAL DISTRIBUTIONS OF SPEARMAN'S AND KENDALL'S COEFFICIENTS

(2·1) *The distributions of r_S and r_K*

The experimental sampling made full use of the 25,000 sets of correlated normal deviates referred to in § 1·1. Thus, we had 2500 samples with $n = 10$, 833 with $n = 30$ and 500 with $n = 50$. For each value of n we had samples from nine bivariate normal populations, namely those with $\rho = 0{\cdot}1\,(0{\cdot}1)\,0{\cdot}9$. The samples of 10, 30 and 50 were independent in the sense that the 25,000 cards containing the basic data were re-shuffled between each of the three experiments. The basic calculations for our study were all carried out in the Mathematics Division of the National Physical Laboratory. The samples were formed and ranked on the Division's punched card installation under the supervision of Miss M. U. Thomas. She was responsible, also, for the calculation of all the values of S_S (of equation (1)) and S_F (the numerator on the right-hand side of equation (4)) and for that of P_K (of equation (3)) for samples of size 10. The values of P_K for samples of sizes 30 and 50 were obtained on the Deuce digital computer by Mr T. Vickers and Mr B. W. Munday. An account of the methods used will be given in a later paper; we plan also to print the observed frequency distributions corresponding to the various coefficients.

Comparison of the observed mean values of r_S with the theoretical values of equation (5) and of the means and variances of r_K with equations (7) and (8) is only useful as a check on the representative character of the random samples. This check has been made and passed satisfactorily; the observed values are not reproduced here. Examination of the variance of r_S is however necessary, equation (6) giving an approximation only to the true value.

(2·2) *The variance of r_S*

The Kendall formula (6) does not give the correct values of $1/(n-1)$ and 0 to $\operatorname{var}(r_S)$ when $\rho = 0$ and 1, respectively. A purely empirical adjustment is obtained by substituting $n-1$ for n as divisor and adding a term $+0{\cdot}019785\rho^{12}$ which reduces the variance to zero when $\rho = 1$, so that we have

$$\operatorname{var}(r_S) = \frac{1}{n-1}\{1 - 1{\cdot}563465\rho^2 + 0{\cdot}304743\rho^4 + 0{\cdot}155286\rho^6 + 0{\cdot}061552\rho^8 + 0{\cdot}022099\rho^{10} + 0{\cdot}019785\rho^{12}\}. \qquad (11)$$

Table 1 contains for each of the three sample sizes, (*a*) the estimated variance from equation (11), (*b*) the observed variance from the sampling experiment, (*c*) smoothed values of (*b*) obtained by a rough graphical process. These last values are made use of in § 3·2 below. It will be seen that for $n = 10$, the modified Kendall formula (11) gives values which for $\rho \geqslant 0{\cdot}3$ are consistently smaller than the observed values. The theoretical approximation is also too small, but less noticeably so, when $n = 30$. It seems clear that for small samples $\operatorname{var}(r_S)$ cannot be accurately expressed as the product of a function of n and a function of ρ. Below, when approximating to the variance of $z_S = \tanh^{-1} r_S$ we have therefore used the smoothed observed values of $\operatorname{var}(r_S)$ taken from the third columns of Table 1.

Table 1. *Variance of Spearman's r_S*

ρ	$n = 10$			$n = 30$			$n = 50$		
	From (11)	Obs.	Smoothed obs.	From (11)	Obs.	Smoothed obs.	From (11)	Obs.	Smoothed obs.
0·1	0·1094	0·1061	0·1093	0·0339	0·0334	0·0342	0·0201	0·0192	0·0203
0·2	0·1042	0·1041	0·1055	0·0323	0·0338	0·0336	0·0191	0·0215	0·0200
0·3	0·0958	0·1002	0·1002	0·0297	0·0321	0·0317	0·0176	0·0192	0·0183
0·4	0·0843	0·0916	0·0923	0·0261	0·0263	0·0273	0·0155	0·0160	0·0165
0·5	0·0701	0·0801	0·0805	0·0218	0·0227	0·0225	0·0129	0·0133	0·0135
0·6	0·0539	0·0638	0·0644	0·0167	0·0181	0·0172	0·0099	0·0110	0·0105
0·7	0·0366	0·0443	0·0470	0·0114	0·0117	0·0117	0·0067	0·00604	0·0065
0·8	0·0199	0·0322	0·0303	0·0062	0·00674	0·0067	0·0037	0·00348	0·0035
0·9	0·0062	0·0125	0·0130	0·0019	0·00241	0·0024	0·0011	0·00111	0·0011

3. The transformation of the rank correlation coefficients

(3·1) *The transformation and its justification*

Our object is to find transformations of r_S and r_K which will give variances approximately independent of ρ and will at the same time make the distributions roughly normal. The basic distributions of r_S and r_K become increasingly skew as $|\rho| \to 1$. It is natural that we should consider R. A. Fisher's z transform which proved so successful in the case of the product moment correlation coefficient r_{xy} in normal samples. If we write in general

$$z = \tanh^{-1} r = \tfrac{1}{2} \log_e \frac{1+r}{1-r}, \tag{12}$$

then the cumulants of z may be expanded in series in terms of the cumulants of r. The leading terms of the expansions for the mean and variance of z are given in equations (13) and (14):

$$\mathscr{E}(z) = \tfrac{1}{2} \log_e \frac{1+\bar{r}}{1-\bar{r}} + \frac{\bar{r}\kappa_2(r)}{(1-\bar{r}^2)^2} + \ldots, \tag{13}$$

$$\text{var}\,(z) = \frac{\kappa_2(r)}{(1-\bar{r}^2)^2} + \ldots, \tag{14}$$

where $\bar{r} = \kappa_1(r) = \mathscr{E}(r)$.

The distribution of r depends only on the single parameter ρ and it will be seen that to a first approximation the z transformation may be expected to stabilize the variance of a statistic r if the ratio of var (r) to $(1-\bar{r}^2)^2$ is independent of ρ, or nearly so. We have given these ratios in Table 2, $\bar{r}_S$, $\bar{r}_K$ and var (r_K) being obtained exactly from equations (5), (7) and (8), respectively, and for var (r_S) we have used the smoothed observed values from Table 1. The ratios are least constant for $n = 10$ where, in particular, there is a definite increase for $\rho = 0{\cdot}9$. Further useful comment must await the calculation of $\kappa_3(r)$ and $\kappa_4(r)$ and a fuller study of the expansions for the cumulants of z, but in the meantime we have felt no hesitation in going further with the use of the z transforms.

Table 2. *First approximations to the variance of z_S and z_K*

ρ	var $(r_S)/(1-\bar{r}_S^2)^2$*			var $(r_K)/(1-\bar{r}_K^2)^2$†		
	$n = 10$	$n = 30$	$n = 50$	$n = 10$	$n = 30$	$n = 50$
0·1	0·111	0·035	0·021	0·0618	0·0166	0·00952
0·2	0·112	0·036	0·021	0·0619	0·0166	0·00950
0·3	0·115	0·037	0·022	0·0622	0·0166	0·00947
0·4	0·120	0·037	0·022	0·0627	0·0165	0·00943
0·5	0·124	0·037	0·022	0·0634	0·0165	0·00937
0·6	0·126	0·037	0·023	0·0644	0·0164	0·00930
0·7	0·130	0·038	0·022	0·0662	0·0164	0·00920
0·8	0·141	0·039	0·022	0·0697	0·0164	0·00910
0·9	0·155	0·043	0·022	0·0787	0·0168	0·00905
Mean $\rho = 0·1$–$0·8$	0·1224	0·0370	0·0219	0·06404	0·01649	0·00936

* var (r_S) obtained from smoothing the experimental values.
† var (r_K) is the correct theoretical value.

Frequency tables of the distributions of z_S and z_K have been obtained and the following sections are concerned with comments on the mean values and variances obtained from these tables and with the normality of the distributions.

(3·2) *The mean values of* $z_S = \tanh^{-1} r_S$ *and* $z_K = \tanh^{-1} r_K$

In Table 3 we compare

(*a*) the observed mean values of z_S found from the experimental data;

(*b*) the approximate theoretical value of $\mathscr{E}(z_S)$ given by the first two terms of (13), namely

$$\mathscr{E}(z_S) = \tanh^{-1} \bar{r}_S + \bar{r}_S \operatorname{var}(r_S)/(1-\bar{r}_S^2)^2, \tag{15}$$

where $\bar{r}_S$ is calculated from (5) and var (r_S) is the smoothed observed value already referred to;

(*c*) the second or 'corrective term' from the right-hand side of (15).

Owing to the fact that in a few samples of 10, the rankings of the two variates were in perfect agreement, some values of r_S (and r_K) are unity and the corresponding z_S (and z_K) become infinite.* The means and variances tabled omit these observations which in any case form a very small part of a distribution of 2500 observations. We first, however, made estimates of the mean and variance of z, using the technique for a censored distribution, but the difference in results was not large enough to be of importance. Having regard to the standard errors quoted below the table† it will be seen that the differences between observation and approximate theory are not significant except perhaps in the case of $\rho = 0·9$. The corrective term is of some importance in small samples with large ρ, but is steadily

* This happened in seven cases for $\rho = 0·9$, in three cases for $\rho = 0·8$ and once for $\rho = 0·7$.

† The standard error of $\bar{z}$ is $\sigma_z/\sqrt{N}$, where the averaged values of σ_z given below Table 5 for z_S and Table 6 for z_K have been used and $N = 2500$, 833 and 500, respectively.

reduced in importance as n increases. In the case of the transformed product moment correlation coefficient a similar, if less important, effect is present. Table 4 gives similar results for the mean values of z_K, except that in this case the true values of var (r_K) may be

Table 3. *Mean values of* z_S

ρ	$n = 10$			$n = 30$			$n = 50$		
	Obs.	Approx. theory	Corr. term	Obs.	Approx. theory	Corr. term	Obs.	Approx. theory	Corr. term
0·1	0·094	0·097	0·010	0·097	0·096	0·003	0·096	0·096	0·002
0·2	0·195	0·195	0·019	0·195	0·194	0·007	0·191	0·194	0·004
0·3	0·304	0·299	0·030	0·297	0·296	0·010	0·297	0·296	0·006
0·4	0·416	0·409	0·042	0·410	0·405	0·014	0·406	0·405	0·008
0·5	0·526	0·529	0·055	0·517	0·525	0·017	0·522	0·525	0·010
0·6	0·671	0·665	0·068	0·661	0·661	0·021	0·665	0·663	0·013
0·7	0·842*	0·825	0·082	0·833	0·826	0·025	0·838	0·828	0·014
0·8	1·032*	1·038	0·103	1·051	1·043	0·030	1·056	1·047	0·016
0·9	1·374*	1·361	0·131	1·406	1·389	0·038	1·417	1·399	0·019

Approximate theory $= \tanh^{-1}\bar{r}_S + \bar{r}_S \operatorname{var}(r_S)/(1-\bar{r}_S^2)^2$, using $\bar{r}_S$ from equation (5) and the smoothed observed var (r_S).

Corrective term = second term in expression for approximate theory.

Standard errors of observed means: for $n = 10$ about 0·008; for $n = 30$, 50, about 0·007.

Table 4. *Mean values of* z_K

ρ	$n = 10$			$n = 30$			$n = 50$		
	Obs.	Approx. theory	Corr. term	Obs.	Approx. theory	Corr. term	Obs.	Approx. theory	Corr. term
0·1	0·065	0·068	0·004	0·066	0·065	0·001	0·065	0·064	0·001
0·2	0·135	0·137	0·008	0·131	0·131	0·002	0·128	0·130	0·001
0·3	0·209	0·209	0·012	0·199	0·200	0·003	0·199	0·198	0·002
0·4	0·289	0·285	0·016	0·275	0·273	0·004	0·271	0·271	0·002
0·5	0·361	0·368	0·021	0·346	0·352	0·005	0·346	0·350	0·003
0·6	0·465	0·462	0·026	0·439	0·442	0·007	0·439	0·439	0·004
0·7	0·582*	0·574	0·033	0·551	0·549	0·008	0·550	0·545	0·005
0·8	0·717*	0·719	0·041	0·692	0·688	0·010	0·689	0·684	0·005
0·9	0·961*	0·949	0·056	0·917	0·905	0·012	0·909	0·899	0·006

Approximate theory $= \tanh^{-1}\bar{r}_K + \bar{r}_K \operatorname{var}(r_K)/(1-\bar{r}_K^2)^2$, where $\bar{r}_K$ and var (r_K) are derived from equations (7) and (8).

Corrective term = second term in expression for approximate theory.

Standard errors of observed means: for $n = 10$ about 0·0055; for $n = 30$, 50 about 0·0045.

* Ignoring the few infinite values.

Table 5. *Observed variance and standard deviation of z_S*

ρ	Variance			S.D.		
	$n = 10$	$n = 30$	$n = 50$	$n = 10$	$n = 30$	$n = 50$
0·1	0·1380	0·0365	0·0204	0·371	0·191	0·143
0·2	0·1407	0·0378	0·0239	0·375	0·194	0·155
0·3	0·1473	0·0405	0·0238	0·384	0·201	0·154
0·4	0·1528	0·0374	0·0226	0·391	0·193	0·150
0·5	0·1537	0·0407	0·0230	0·392	0·202	0·152
0·6	0·1507	0·0389	0·0246	0·388	0·197	0·157
0·7	0·1423*	0·0380	0·0213	0·377	0·195	0·146
0·8	0·1643*	0·0409	0·0227	0·405	0·202	0·151
0·9	0·1700*	0·0465	0·0235	0·412	0·216	0·153
Mean, $\rho = 0{\cdot}1$–$0{\cdot}8$	0·14872	0·03884	0·02279	0·385	0·197	0·151
			$\frac{1{\cdot}0296}{\sqrt{(n-3)}}$	0·389	0·198	0·150

Standard errors of S.D.: $n = 10$, 0·0055; $n = 30$, 0·0049; $n = 50$, 0·0047.

Table 6. *Observed variance and standard deviation of z_K*

ρ	Variance			S.D.		
	$n = 10$	$n = 30$	$n = 50$	$n = 10$	$n = 30$	$n = 50$
0·1	0·06830	0·01700	0·00933	0·2613	0·1304	0·0966
0·2	0·06884	0·01758	0·01074	0·2624	0·1327	0·1036
0·3	0·07290	0·01830	0·01049	0·2700	0·1353	0·1024
0·4	0·07446	0·01639	0·00991	0·2729	0·1280	0·0996
0·5	0·07443	0·01712	0·00966	0·2728	0·1308	0·0983
0·6	0·07384	0·01628	0·00985	0·2717	0·1276	0·0992
0·7	0·06949*	0·01514	0·00822	0·2636	0·1230	0·0907
0·8	0·08126*	0·01551	0·00824	0·2851	0·1245	0·0908
0·9	0·08910*	0·01712	0·00780	0·2985	0·1308	0·0883
Mean, $\rho = 0{\cdot}1$–$0{\cdot}8$	0·07294	0·01667	0·00956	0·2700	0·1290	0·0977
			$\frac{0{\cdot}6611}{\sqrt{(n-4)}}$	0·2699	0·1297	0·0975

Standard errors of S.D.: $n = 10$, 0·0038; $n = 30$, 0·0032; $n = 50$, 0·0031.

* Ignoring the few infinite values.

derived from equation (8). Again, the differences between observation and the approximation appear only to be significant when $\rho = 0{\cdot}9$. The corrective term is important for $n = 10$ and ρ large; it is smaller in proportion than the corresponding term in the approximation to mean z_S.

(3·3) *The variances and standard deviations of z_S and z_K*

Tables 5 and 6 contain the observed variances and standard deviations for the transformed variables*. Comparison with Table 2 shows that the first term in the expansions for $\text{var}(z_S)$ and $\text{var}(z_K)$ is definitely not adequate when $n = 10$ and still somewhat in defect for the larger samples. These points are brought out by a comparison of the mean values given at the bottom of the tables for the eight cases $\rho = 0{\cdot}1$ to $0{\cdot}8$. Apart from the extreme case with $\rho = 0{\cdot}9$, the change in the variance of z with ρ is not very great. We shall not attempt now to discuss the changes further. The figures, however, suggest that for most practical purposes if $\rho \leqslant 0{\cdot}8$ it will be justifiable to assume a constant variance for z, for any given sample size not greatly exceeding 50. The expressions given below are not, however, to be regarded as asymptotic results.

Assuming that we may use the observed mean values given at the bottom of Tables 5 and 6 we may then look for a general empirical expression for the variance of the form

$$\text{var}(z) = a/(n-b),$$

where b is an integer. We suggest the use of the following:

For Spearman's coefficient

$$\text{var}(z_S) = \frac{1{\cdot}060}{n-3}, \quad \sigma_z = \frac{1{\cdot}03}{\sqrt{(n-3)}}.$$

For Kendall's coefficient

$$\text{var}(z_K) = \frac{0{\cdot}437}{n-4}, \quad \sigma_z = \frac{0{\cdot}66}{\sqrt{(n-4)}}.$$

The resulting approximations for the standard deviations of z_S and z_K when $n = 10$, 30 and 50 are given at the bottom of the right-hand side of Tables 5 and 6 where they may be compared with the individual sampled values and the means of the latter for $\rho = 0{\cdot}1$ to $0{\cdot}8$. We think that except for $\rho > 0{\cdot}8$, the approximation can be safely used in tests of significance for $10 \leqslant n \leqslant 50$, provided, of course, that the underlying conditions discussed in § 1·3 are applicable to the data.

4. Normality of the z distributions; preliminary comments

In the case of $n = 10$ three difficulties arise in examining the fit of a normal curve to the experimental distributions. In the first place, as mentioned above, in a few samples there was complete agreement between the two rankings so that r_S and r_K were unity and z_S and z_K were consequently infinite. These observations have to be omitted in calculating the moments of z; alternatively, moments could be estimated using the technique for dealing with truncated observations. Secondly, while possible values for r_S and r_K are equally spaced, the possible values of z_S and z_K occur at intervals which increase with the z value. For $n = 10$, where the number of permissible values is relatively small, it is a little difficult to know what criterion of normality to adopt. Finally, the distributions of r_S exhibit, particularly for low values of ρ, the 'saw-edged' character noted by Kendall (1948, p. 47) in the case of independence. These factors all make it difficult to know how to

* For $n = 30$, 50 the variances tabled are m_2, but for $n = 10$ they are $k_2 = m_2 N/(N-1)$.

assess the importance of excessive values of χ^2, when comparing the observed distributions with fitted normal curves. Although we have not yet available all the values of $\beta_2(z)$, it appears, as in the case of the z transform of the product moment correlation coefficient, that the z distributions are somewhat leptokurtic ($\beta_2 > 3$).

For $n = 30$ and 50 we have fitted a certain number of normal curves to the z distributions. The result of applying the χ^2 test for goodness-of-fit is summarized in Table 7. Apart from three values of χ^2 which are over 30, the fits appear very reasonable. It is clear that the matter needs further investigation, but we doubt whether even in samples as small as 10, the assumption of a normal z distribution will lead to any serious misinterpretation of a significance test.

Table 7. *Normal curve fits to observed distributions of z_S and z_K*

ρ	z_S; $n = 30$		z_S; $n = 50$		z_K; $n = 30$		z_K; $n = 50$	
	χ^2	D.F.	χ^2	D.F.	χ^2	D.F.	χ^2	D.F.
0·1	21·1	21	16·3	19	—	—	—	—
0·2	32·9	22	20·6	20	—	—	—	—
0·3	37·8	22	20·1	20	—	—	—	—
0·4	23·7	21	23·1	20	—	—	—	—
0·5	28·7	22	13·1	20	22·1	21	12·9	21
0·6	11·1	21	17·1	20	—	—	—	—
0·7	16·6	21	29·7	19	—	—	—	—
0·8	21·9	22	15·6	19	26·0	20	14·7	21
0·9	23·0	23	15·9	20	34·8	21	22·3	19
Total	216·8	195	171·5	177	82·9	62	49·9	61

5. THE SENSITIVITY OF THE CORRELATION MEASURES TO CHANGES IN ρ

In broad terms the power of discrimination of any one of the possible correlation measures depends upon the rapidity with which its sampling distributions draw clear of one another as the population ρ changes. If, for example, for a given value of n, the distribution of r_S for $\rho = 0{\cdot}2$ does not sensibly overlap the distribution for $\rho = 0{\cdot}8$, then if a single sample of n is drawn from each population a test of significance will always establish a difference in population ρ values. The amount of overlap can of course be seen most directly in the distributions of r_S and r_K (or S_S and P_K) which we hope to publish later.

If the distributions of the z's were normal with a standard deviation σ_z which is fixed for a given sample size, the efficiency of discrimination would depend on the way in which the scale of mean z expressed in standard measure (i.e. $\mathscr{E}(z)/\sigma_z$) opened out as ρ is increased from 0 to 1. Without assuming a constant σ_z, we can obtain a rough measure of local sensitivity by calculating the ratios $(\bar{z}_2 - \bar{z}_1)/\sqrt{(s^2_{\bar{z}_1} + s^2_{\bar{z}_2})}$ of

(*a*) the differences between pairs of consecutive observed means given in Table 3 (or 4), to

(*b*) the square roots of the sum of the corresponding pair of observed variances from Table 5 (or 6).

These ratios are given in Table 8 for both Spearman's and Kendall's z. We have also given corresponding ratios for the product moment correlation coefficient, taking $\mathscr{E}(z)$ and σ_z^2 from the full Fisher expansions as corrected by Gayen (1951, p. 236). Having regard to sampling fluctuations, it is clear that we cannot establish any difference in sensitivity between the two rank coefficients for $n = 10$. At $n = 30$ and 50 and for $\rho > 0{\cdot}6$ the ratio for z_K is consistently larger than for z_S, which suggests a possible advantage for Kendall's coefficient. More detailed examination of this point is however needed. It will be noted, as expected, that the product moment coefficient is throughout more sensitive to changes in ρ than either of the rank coefficients. In all cases for a given difference in ρ, the power of discrimination increases with ρ.

Table 8. *Sensitivity ratios* $(\bar{z}_1 - \bar{z}_2)/\sqrt{(s_{z_1}^2 + s_{z_2}^2)}$ *for different coefficients*

ρ_1, ρ_2	$n = 10$			$n = 30$			$n = 50$		
	Product moment	Spearman	Kendall	Product moment	Spearman	Kendall	Product moment	Spearman	Kendall
0·1, 0·2	**0·205**	0·191	0·188	**0·383**	0·359	0·352	**0·501**	0·452	0·447
0·2, 0·3	**0·214**	0·202	0·198	**0·399**	0·362	0·359	**0·523**	0·489	0·484
0·3, 0·4	**0·228**	0·206	0·208	**0·427**	0·405	0·406	**0·558**	0·503	0·508
0·4, 0·5	**0·250**	0·198	0·188	**0·469**	0·384	0·387	**0·615**	0·542	0·538
0·5, 0·6	**0·286**	0·263	0·270	**0·537**	0·508	0·508	**0·704**	0·655	0·664
0·6, 0·7	**0·345**	0·315	0·308	**0·649**	0·622	0·636	**0·851**	0·809	0·828
0·7, 0·8	**0·456**	0·345	0·348	**0·861**	0·776	0·805	**1·13**	1·04	1·08
0·8, 0·9	**0·732**	0·592	0·592	**1·39**	1·20	1·24	**1·82**	1·68	1·74

Concluding remarks

Besides putting on record the basic sampling distributions we hope in a further paper to carry our investigations further in a number of directions, in particular to give parallel results for the coefficient r_F of equation (4).

We should like to express our great indebtedness to Miss M. U. Thomas and Mr T. Vickers whose work has already been mentioned, to Mrs Esmé Hill, formerly of the Statistical Advisory Unit, Ministry of Supply, and to Mrs Maxine Merrington and Miss Janet Hall of University College London. Finally, we should like to say how much we owe to the co-operation of the Mathematics Division of the National Physical Laboratory for the facilities which made this investigation possible.

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THE TWO-SAMPLE t-TEST BASED ON RANGE

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1. Introduction

A test for the significance of a difference in means, using an estimate of standard deviation based on sample range, was suggested by Lord (1947) as a quick substitute for Student's t-test in the case where the two samples are of equal size. Thus, in Table 10 of his paper Lord gives six significance levels for the ratio $|\bar{x}_1-\bar{x}_2|/\frac{1}{2}(w_1+w_2)$, where $\bar{x}_1$, $\bar{x}_2$ are the means and w_1, w_2 the ranges in samples of size n from two normal populations having a common standard deviation, σ. The null hypothesis is that the population means, μ_1, μ_2 are equal.

This test has been in considerable use as a quick method of detecting differences in means, particularly in the industrial field. In a discussion which took place at a weekend conference of the Industrial Applications Section of the Royal Statistical Society, held in July 1956, speakers asked whether it would be possible to provide a similar quick test for the case where the sample sizes n_1, n_2 were not equal. The present paper gives an answer to this question.

When $n_1 \neq n_2$, a number of points need special consideration.

(*a*) The most efficient range estimate of σ is no longer based on the unweighted mean of the two sample ranges. If for practical convenience we decide to use $\theta = w_1+w_2$ as our estimating statistic rather than $\phi = w_1+f(n_1,n_2)\,w_2$ (where f is an appropriate weighting function), it is desirable to know what loss of power is involved. This will be considered in § 4, where it is shown that the loss is small.

(*b*) In Table 2 we give, however, two functions which enable the more accurate estimate of σ based on ϕ to be obtained if desired.

(*c*) With $n_1 \neq n_2$ there is no longer any point in using Lord's mean range and we shall take as our test ratio

$$u = \frac{|\bar{x}_1-\bar{x}_2|}{w_1+w_2}. \tag{1}$$

Table 1 printed at the end of this paper gives values of u corresponding to four different significance levels, α, used in a two-tailed test.

(*d*) We have provided a table for all combinations of n_1, n_2 in the range 2 to 20. Computation by quadrature as used by Lord in the case $n_1 = n_2$ would now be very heavy and it was therefore decided to use an approximation. We have adopted the type of approximation first suggested by Patnaik (1950) and have taken w_1+w_2 to be distributed as a multiple of χ with appropriate 'equivalent' degrees of freedom ν depending on n_1 and n_2. Following this approach it is possible to represent the distribution of the ratio u of equation (1) in terms of Student's t with modified degrees of freedom which are less than n_1+n_2. The accuracy of this approximation is examined in § 5. We shall first give two examples illustrating the use of Table 1.

2. Illustrations

Example (i). Two operators, A and B, make determinations of the percentage of ammonia in plant gas, with the following results:

Operator A	39	35	43	32	36	48	33	33
Operator B	43	44	56	63	46			

From these figures can it be said that the two operators are consistently measuring the same thing?

The data give

$$\bar{x}_1 = 37{\cdot}375, \quad w_1 = 16, \quad \sum_i (x_{1i}-\bar{x}_1)^2 = 221{\cdot}875;$$

$$\bar{x}_2 = 50{\cdot}4, \quad w_2 = 20, \quad \sum_i (x_{2i}-\bar{x}_2)^2 = 305{\cdot}2.$$

Hence

$$u = \frac{50{\cdot}4-37{\cdot}375}{16+20} = 0{\cdot}362,$$

and from Table 1 with $n_1 = 5$ and $n_2 = 8$, u is just beyond the 1 % level of significance, remembering that we are using a two-tailed test here. The usual form of t-test gives

$$t = \frac{50{\cdot}4 - 37{\cdot}375}{\sqrt{\left(\frac{1}{8}+\frac{1}{5}\right)\left(\frac{527{\cdot}075}{11}\right)}} = 3{\cdot}301.$$

The degrees of freedom, ν, are 11 and t is again just beyond the 1 % level of significance. Hence, with both tests we would have detected a real difference between the two operators.

Example (ii). Twelve children aged 12 years to 12 years 3 months were selected at random from a school and fed on a special diet of pasteurized milk for 4 months. The gains in weight, in ounces, of the twelve children over the 4-month period were

$$7, \quad 17, \quad 53, \quad -2, \quad 27, \quad 41, \quad 37, \quad 35, \quad 10, \quad 12, \quad 9, \quad 38.$$

Eight children of similar age were selected as 'controls' and not given the diet. The gains in weight were

$$10, \quad 0, \quad 29, \quad 11, \quad -21, \quad 25, \quad 19, \quad -19.$$

From these data it is desired to investigate whether the pasteurized milk diet results in a greater increase in weight over the 4-month period. The figures give

$$\bar{x}_1 = 23{\cdot}667, \quad w_1 = 55, \quad \sum_i (x_{1i} - \bar{x}_1)^2 = 3202{\cdot}7;$$

$$\bar{x}_2 = 6{\cdot}750, \quad w_2 = 50, \quad \sum_i (x_{2i} - \bar{x}_2)^2 = 2485{\cdot}5.$$

Hence

$$u = \frac{23{\cdot}667 - 6{\cdot}750}{50 + 55} = 0{\cdot}161.$$

Using a one-tailed test, and entering Table 1 with $n_1 = 8$, $n_2 = 12$, it is seen that u falls just beyond the $2\frac{1}{2}$ % significance point.

If the t-test is used, we have $t = 2{\cdot}085$; with $\nu = 18$ this is a value not quite reaching the $2\frac{1}{2}$ % level of significance. Both tests therefore show that there is an increase in mean weight associated with the pasteurized milk diet and that approximately this is significant at the $2\frac{1}{2}$ % level.

3. The range estimator of standard deviation

When the samples are of unequal size, the unweighted sum

$$\theta = w_1 + w_2 \tag{2}$$

does not provide the best estimate of the assumed common population standard deviation, σ. It may be shown (see David, 1951) that

$$\phi = w_1 + f(n_1, n_2)\, w_2 \tag{3}$$

will provide the range estimate with minimum coefficient of variation, where

$$f(n_1, n_2) = \frac{d_{n_2}}{V_{n_2}} \times \frac{V_{n_1}}{d_{n_1}}, \tag{4}$$

d_n and V_n being the mean and variance of range in a random sample of size n from a normal population having unit standard deviation. Values of d_n, V_n and their ratio are given by Pearson & Hartley (1954, Table 20). The unbiased estimate of σ is then

$$g = \frac{w_1 + f(n_1, n_2)\, w_2}{d_{n_1} + f(n_1, n_2)\, d_{n_2}}. \tag{5}$$

As we shall proceed to show, the gain from using ϕ rather than θ does not appear worthwhile in a test of significance. As there may, however, be situations where the more accurate range estimate of σ is preferred, we give in Table 2 at the end of this paper values of $f(n_1, n_2)$ from equation (4) and of the denominator $d_{n_1} + f(n_1, n_2)\, d_{n_2}$ of estimator g of equation (5). The values of n_1, n_2 go from 2 to 20, n_2 corresponding to the larger of the two samples.

4. Justification for using the unweighted sum of ranges

We start by comparing in Table 3 the coefficients of variation of θ and ϕ for certain combinations of n_1 and n_2. It will be seen that only when the samples are of very unequal size is there likely to be any marked loss of efficiency. A more direct method of studying the loss of efficiency resulting from the use of θ instead of ϕ in a test of significance is to compare the power functions of the corresponding tests.

Table 3. *Ratio of coefficients of variation of θ and ϕ*

n_2 \ n_1	3	5	7	9
3	1·000	—	—	—
6	1·030	1·002	—	—
9	1·066	1·018	1·003	1·000
12	1·097	1·036	1·013	1·003
16	1·131	1·060	1·029	1·013
20	1·158	1·080	1·043	1·024

Table 4. *Power functions of t and modified t-tests*

Sample sizes	Type of test	Values of $(\mu_1-\mu_2)/\sigma$				
		1	2	3	4	5
$n_1 = 6$	R.M.S.	0·244	0·602	0·886	0·984	0·999
$n_2 = 10$	w_1+fw_2	0·242	0·596	0·882	0·983	0·999
	w_1+w_2	0·241	0·593	0·879	0·982	0·999
$n_1 = 3$	R.M.S.	0·249	0·614	0·895	0·987	0·999
$n_2 = 20$	w_1+fw_2	0·245	0·605	0·889	0·985	0·999
	w_1+w_2	0·241	0·593	0·879	0·982	0·999
$n_1 = 2$	R.M.S.	0·210	0·504	0·791	0·944	0·991
$n_2 = 4$	w_1+fw_2	0·207	0·492	0·775	0·935	0·989
	w_1+w_2	0·204	0·482	0·767	0·929	0·986

As will be suggested below a good approximation to the distribution of θ, and presumably also of ϕ, may be obtained from the χ-distribution. From this it follows that the distribution of the test ratio u of equation (1) and that of a similar ratio using ϕ instead of θ will follow t-distributions if the population means μ_1 and μ_2 are equal and non-central t-distributions if $\mu_1 \neq \mu_2$. The 'equivalent' degrees of freedom using ϕ will be less than the standard value $\nu = n_1+n_2-2$ of Student's test, and the degrees of freedom using θ will be somewhat less than those using ϕ. What this means in terms of power may be found using the tables of the power function of the t-test given by Neyman (1935) for the one-tailed test. Results of this comparison for a 5% significance level test are shown in Table 4, using the appropriate 'equivalent' degrees of freedom for the θ and ϕ estimates in the manner outlined in the following section. It is seen that while both forms of test based on range are less powerful than the standard test using the root-mean-square estimate of σ, the difference between the θ and ϕ forms is not large.

We conclude that if a range test is required, then the simpler form involving the straight sum $\theta = w_1+w_2$ may be used since the amount of power sacrificed is negligible.

5. CALCULATION OF SIGNIFICANT POINTS

Before making extensive use of Patnaik's χ-approximation it was decided to test its accuracy in one case, with $n_1 = 4$, $n_2 = 16$, by also making a full evaluation by quadrature.

(i) *Quadrature*

It is not necessary to describe the method used in detail. Use was made of the values of the ordinates of the distributions of range required, kindly supplied by Mr Lord, who had previously derived them from the basic 5-decimal-place tables of the probability integral of range used by Pearson & Hartley (1942). As a result, we could find the exact significance points of a normal variable divided by an estimate of its standard error based on the sum of two independent ranges.

(ii) *The χ-approximation*

Patnaik's method leads us to identify the first two moments of $(w_1+w_2)/\sigma$, where σ is the population standard deviation, with the first two moments of $c\chi/\sqrt{\nu}$ where χ has ν degrees of freedom. The first two moments of $(w_1+w_2)/\sigma$ are

$$M = d_{n_1}+d_{n_2}, \quad V = V_{n_1}+V_{n_2} \tag{6}$$

and these are equated to those of $c\chi/\sqrt{\nu}$ to give

$$M = \frac{c}{\sqrt{\nu}}\sqrt{2}\,\Gamma\left(\frac{\nu+1}{2}\right)\Big/\Gamma\left(\frac{\nu}{2}\right), \tag{7}$$

$$V = \frac{c^2}{\nu}\left[\nu - 2\left\{\Gamma\left(\frac{\nu+1}{2}\right)\Big/\Gamma\left(\frac{\nu}{2}\right)\right\}^2\right]. \tag{8}$$

Expanding the gamma functions by Stirling's formula we obtain

$$k = \frac{V}{M^2} = \frac{1}{2\nu} + \frac{1}{8\nu^2} - \frac{1}{16\nu^3} + \ldots. \tag{9}$$

As a first approximation

$$\nu^{-1} = -2+2\sqrt{(1+2k)} \tag{10}$$

and a better approximation leads to

$$\nu^{-1} = -2+2\sqrt{(1+2j)}, \tag{11}$$

where $j = k+1/(16\nu^3)$ the value of ν from (10) being used for j. The expression for c to be the same order of accuracy is

$$c = M\left\{1+\frac{1}{4\nu}+\frac{1}{32\nu^2}-\frac{5}{128\nu^3}\right\}. \tag{12}$$

Hence, if x is a unit normal variable the distribution of $x(d_{n_1}+d_{n_2})/(w_1+w_2)$ can be approximated to by Mt/c, where the t-distribution has the 'equivalent' degrees of freedom from (11) above. The percentage points can be found by interpolating for fractional ν in a table of the percentage points of the t-distribution and multiplying by the appropriate factor.

For the case where $n_1 = 4$ and $n_2 = 16$ the percentage points calculated by the two methods are:

α (two-tailed)	Quadrature	χ-approximation
0·10	1·744	1·746
0·05	2·130	2·136
0·02	2·615	2·628
0·01	2·971	2·995

A comparison of these results with those given by Patnaik (1950, p. 81) for equal sample sizes shows that the χ-approximation is as good for the unequal sample sizes considered here as it was for the equal sample sizes. With this check it was felt safe to proceed with the calculation of the percentage points using the χ-approximation.

6. Calculation of table

For each of the 190 pairs of sample sizes, where n_1 and n_2 go from 2 to 20, the following quantities were calculated:

(i) M and V from d_n and V_n as in (6),

(ii) $1/\nu$ from (10) and (11),

(iii) $c' = c/M$ from (12).

For each value of ν found from (ii) the percentage points of t for $\alpha = 0{\cdot}10$, $0{\cdot}05$, $0{\cdot}02$ and $0{\cdot}01$ were obtained by interpolation in the t tables. Call these values $t_{\nu,\alpha}$. The required percentage point will then be

$$u_\alpha = \frac{t_{\nu,\alpha}\sqrt{\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}{c'(d_{n_1}+d_{n_2})}. \tag{13}$$

A subsidiary table of $\sqrt{(n_1^{-1}+n_2^{-1})}$ was computed and u_α finally obtained from (13). The values are given in Table 1 to three places of decimals and it is thought that even in the worst cases, that is when both samples are small, or the samples are very unequal in size, the error should not exceed 3 in the third decimal. As a check the values were differenced for n_2 given n_1 and for n_1 given n_2, and in the particular case where n_1 equals n_2 the values can be checked against the values given by Lord (1947, Table 10, p. 66). They should be just half Lord's values since he used $\frac{1}{2}(w_1+w_2)$ for his denominator in u.

We have not considered in detail modifications of this test which could be obtained, for example, by splitting up the sample ranges into the ranges of smaller subsamples. If $n_1 = 8$ and $n_2 = 16$ it might be better to estimate the variability by splitting the second sample up randomly into two smaller subsamples, each of eight, giving us three ranges w_1, w_2' and w_2''. The expected gain can be gauged by finding the power function for a 5% significance level test as in Table 3. For the three ranges $w_1+w_2'+w_2''$ we have

$$M = 3d_8 = 8{\cdot}54160, \quad V = 3V_8 = 2{\cdot}01639,$$

and using equations (10) and (11) ν is found to be 18·33. If the two ranges are used in the form w_1+w_2 as in this paper $\nu = 16{\cdot}72$, whilst if the root-mean-square estimate of standard deviation is used $\nu = 22$. The powers from Neyman's tables are

μ/σ	1	2	3	4	5
R.M.S.	0·250	0·615	0·896	0·987	0·999
$w_1+w_2'+w_2''$	0·248	0·611	0·893	0·986	0·999
w_1+w_2	0·247	0·607	0·891	0·986	0·999

The extra gain by splitting into subsamples thus seems to be very small and would not appear to be worth the very much greater complexity which such a modification would introduce into the test.

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Table 1. *Values of* $u = |\bar{x}_1 - \bar{x}_2|/(w_1 + w_2)$ *exceeded with probability* α
(*for a single-tailed test* α *must be halved*)

n_1	n_2	Probability (α)				n_1	n_2	Probability (α)			
		0·10	0·05	0·02	0·01			0·10	0·05	0·02	0·01
2	2	1·161	1·714	2·776	3·958	4	16	0·175	0·213	0·263	0·299
	3	0·693	0·915	1·255	1·557		17	0·172	0·210	0·258	0·293
	4	0·556	0·732	1·002	1·242		18	0·169	0·206	0·253	0·288
	5	0·478	0·619	0·827	1·008		19	0·166	0·203	0·249	0·283
	6	0·429	0·549	0·721	0·865		20	0·164	0·200	0·246	0·279
	7	0·396	0·502	0·652	0·776	5	5	0·247	0·307	0·387	0·450
	8	0·372	0·469	0·603	0·713		6	0·224	0·277	0·347	0·402
	9	0·353	0·443	0·567	0·666		7	0·208	0·256	0·319	0·368
	10	0·338	0·423	0·538	0·630		8	0·195	0·240	0·299	0·343
	11	0·326	0·407	0·515	0·601		9	0·186	0·228	0·282	0·323
	12	0·316	0·393	0·496	0·557		10	0·178	0·218	0·270	0·309
	13	0·307	0·382	0·480	0·557		11	0·172	0·210	0·260	0·296
	14	0·300	0·372	0·467	0·541		12	0·167	0·204	0·251	0·286
	15	0·294	0·363	0·455	0·526		13	0·162	0·198	0·244	0·277
	16	0·287	0·356	0·445	0·513		14	0·158	0·193	0·237	0·270
	17	0·282	0·349	0·436	0·502		15	0·155	0·189	0·232	0·263
	18	0·278	0·343	0·428	0·492		16	0·152	0·185	0·227	0·257
	19	0·274	0·338	0·420	0·483		17	0·149	0·182	0·222	0·252
	20	0·270	0·333	0·414	0·475		18	0·147	0·179	0·218	0·248
3	3	0·487	0·635	0·860	1·050		19	0·144	0·176	0·215	0·244
	4	0·398	0·511	0·663	0·814		20	0·142	0·173	0·212	0·240
	5	0·339	0·429	0·556	0·660	6	6	0·203	0·250	0·312	0·359
	6	0·311	0·391	0·501	0·590		7	0·188	0·240	0·287	0·329
	7	0·288	0·360	0·458	0·536		8	0·177	0·217	0·268	0·307
	8	0·271	0·338	0·427	0·498		9	0·168	0·206	0·254	0·289
	9	0·258	0·321	0·404	0·469		10	0·161	0·197	0·242	0·276
	10	0·248	0·307	0·385	0·446		11	0·155	0·189	0·233	0·265
	11	0·240	0·296	0·370	0·427		12	0·150	0·183	0·225	0·255
	12	0·232	0·287	0·358	0·412		13	0·146	0·178	0·218	0·247
	13	0·226	0·279	0·347	0·399		14	0·142	0·173	0·212	0·241
	14	0·221	0·272	0·338	0·388		15	0·139	0·169	0·207	0·235
	15	0·216	0·266	0·330	0·378		16	0·136	0·166	0·203	0·229
	16	0·212	0·261	0·323	0·370		17	0·134	0·163	0·199	0·225
	17	0·209	0·256	0·317	0·362		18	0·131	0·160	0·195	0·221
	18	0·205	0·252	0·311	0·356		19	0·129	0·157	0·192	0·217
	19	0·202	0·248	0·306	0·350		20	0·128	0·155	0·189	0·214
	20	0·200	0·245	0·302	0·344	7	7	0·174	0·213	0·263	0·301
4	4	0·322	0·407	0·526	0·620		8	0·163	0·200	0·246	0·281
	5	0·282	0·353	0·450	0·528		9	0·155	0·189	0·233	0·265
	6	0·256	0·319	0·403	0·469		10	0·148	0·181	0·222	0·252
	7	0·237	0·294	0·370	0·429		11	0·143	0·174	0·213	0·242
	8	0·224	0·276	0·346	0·399		12	0·138	0·168	0·206	0·233
	9	0·213	0·263	0·327	0·377		13	0·134	0·163	0·199	0·226
	10	0·204	0·252	0·313	0·359		14	0·131	0·159	0·194	0·220
	11	0·197	0·242	0·301	0·345		15	0·128	0·155	0·189	0·214
	12	0·191	0·235	0·291	0·333		16	0·125	0·152	0·185	0·209
	13	0·186	0·228	0·282	0·322		17	0·123	0·149	0·181	0·205
	14	0·182	0·223	0·275	0·314		18	0·121	0·146	0·178	0·201
	15	0·178	0·218	0·268	0·306		19	0·119	0·144	0·175	0·198
							20	0·117	0·142	0·172	0·195

Table 1 (*continued*)

n_1	n_2	Probability (α)				n_1	n_2	Probability (α)			
		0·10	0·05	0·02	0·01			0·10	0·05	0·02	0·01
8	8	0·153	0·187	0·231	0·262	12	12	0·107	0·130	0·158	0·178
	9	0·145	0·177	0·217	0·247		13	0·104	0·126	0·153	0·172
	10	0·139	0·169	0·207	0·235		14	0·101	0·122	0·148	0·167
	11	0·133	0·162	0·199	0·225		15	0·098	0·119	0·144	0·162
	12	0·129	0·157	0·192	0·217		16	0·096	0·116	0·140	0·158
	13	0·125	0·152	0·186	0·210		17	0·094	0·113	0·137	0·154
	14	0·122	0·148	0·180	0·204		18	0·092	0·111	0·134	0·151
	15	0·119	0·144	0·176	0·199		19	0·090	0·109	0·132	0·149
	16	0·116	0·141	0·172	0·194		20	0·089	0·107	0·130	0·146
	17	0·114	0·138	0·168	0·190	13	13	0·100	0·121	0·147	0·166
	18	0·112	0·136	0·165	0·186		14	0·097	0·118	0·143	0·161
	19	0·110	0·134	0·162	0·183		15	0·095	0·115	0·139	0·156
	20	0·109	0·132	0·160	0·180		16	0·092	0·112	0·135	0·152
9	9	0·137	0·167	0·205	0·233		17	0·090	0·109	0·132	0·149
	10	0·131	0·160	0·195	0·221		18	0·089	0·107	0·130	0·146
	11	0·126	0·153	0·187	0·212		19	0·087	0·105	0·127	0·143
	12	0·122	0·148	0·180	0·204		20	0·086	0·103	0·125	0·140
	13	0·118	0·143	0·175	0·197	14	14	0·094	0·114	0·138	0·156
	14	0·115	0·139	0·170	0·192		15	0·092	0·111	0·135	0·151
	15	0·112	0·136	0·165	0·187		16	0·090	0·108	0·131	0·147
	16	0·110	0·133	0·162	0·182		17	0·088	0·106	0·128	0·144
	17	0·107	0·130	0·158	0·178		18	0·086	0·104	0·125	0·141
	18	0·106	0·128	0·155	0·175		19	0·084	0·102	0·123	0·138
	19	0·104	0·126	0·152	0·172		20	0·083	0·101	0·121	0·135
	20	0·102	0·124	0·150	0·169	15	15	0·089	0·108	0·131	0·147
10	10	0·125	0·152	0·186	0·210		16	0·087	0·105	0·127	0·143
	11	0·120	0·146	0·178	0·201		17	0·085	0·103	0·124	0·140
	12	0·116	0·141	0·171	0·194		18	0·083	0·101	0·122	0·137
	13	0·112	0·136	0·166	0·187		19	0·082	0·099	0·119	0·134
	14	0·109	0·133	0·161	0·182		20	0·080	0·097	0·117	0·131
	15	0·107	0·129	0·157	0·177	16	16	0·085	0·103	0·124	0·139
	16	0·104	0·126	0·153	0·173		17	0·083	0·100	0·121	0·136
	17	0·102	0·124	0·150	0·169		18	0·081	0·098	0·118	0·133
	18	0·100	0·121	0·147	0·165		19	0·080	0·096	0·116	0·130
	19	0·098	0·119	0·144	0·162		20	0·078	0·094	0·114	0·128
	20	0·097	0·117	0·142	0·160	17	17	0·081	0·098	0·118	0·132
11	11	0·115	0·140	0·170	0·193		18	0·079	0·096	0·115	0·130
	12	0·111	0·135	0·164	0·185		19	0·078	0·094	0·113	0·127
	13	0·108	0·131	0·159	0·179		20	0·076	0·092	0·111	0·124
	14	0·105	0·127	0·154	0·174	18	18	0·077	0·093	0·113	0·126
	15	0·102	0·123	0·150	0·169		19	0·076	0·092	0·110	0·124
	16	0·100	0·121	0·146	0·165		20	0·074	0·090	0·108	0·121
	17	0·098	0·118	0·143	0·161	19	19	0·074	0·090	0·108	0·121
	18	0·096	0·116	0·140	0·158		20	0·073	0·088	0·106	0·119
	19	0·094	0·114	0·138	0·155						
	20	0·092	0·112	0·135	0·152	20	20	0·071	0·086	0·104	0·116

Table 2. *To assist in estimating a standard deviation from the range estimator*
$g=(w_1+f(n_1, n_2)\,w_2)/(d_{n_1}+f(n_1, n_2)\,d_{n_2})$

n_1	n_2	f	$d_{n_1}+fd_{n_2}$	n_1	n_2	f	$d_{n_1}+fd_{n_2}$	n_1	n_2	f	$d_{n_1}+fd_{n_2}$
2	3	1·378	3·461	5	13	1·804	8·344	10	11	1·057	6·431
	4	1·713	4·655		14	1·878	8·724		12	1·110	6·694
	5	2·009	5·801		15	1·949	9·093		13	1·160	6·947
									14	1·208	7·192
	6	2·267	6·873		16	2·016	9·446		15	1·253	7·427
	7	2·512	7·920		17	2·080	9·789				
	8	2·731	8·903		18	2·142	10·123		16	1·297	7·658
	9	2·931	9·833		19	2·201	10·445		17	1·338	7·878
	10	3·117	10·721		20	2·258	10·760		18	1·377	8·089
									19	1·416	8·301
	11	3·298	11·592	6	7	1·105	5·522		20	1·452	8·500
	12	3·465	12·419		8	1·202	5·956				
	13	3·620	13·204		9	1·291	6·368	11	12	1·050	6·594
	14	3·768	13·965		10	1·375	6·766		13	1·098	6·836
	15	3·910	14·703						14	1·143	7·067
					11	1·453	7·144		15	1·186	7·291
	16	4·045	15·415		12	1·526	7·506				
	17	4·174	16·104		13	1·595	7·855		16	1·227	7·507
	18	4·296	16·766		14	1·661	8·193		17	1·266	7·715
	19	4·418	17·426		15	1·723	8·516		18	1·303	7·916
	20	4·528	18·040						19	1·340	8·116
					16	1·782	8·828		20	1·374	8·305
3	4	1·240	4·246		17	1·839	9·132				
	5	1·455	5·077		18	1·893	9·425	12	13	1·045	6·744
					19	1·946	9·713		14	1·088	6·965
	6	1·641	5·852		20	1·996	9·989		15	1·129	7·178
	7	1·818	6·610								
	8	1·977	7·322	7	8	1·087	5·799		16	1·168	7·383
	9	2·122	7·995		9	1·168	6·173		17	1·205	7·581
	10	2·257	8·639		10	1·244	6·532		18	1·241	7·775
									19	1·275	7·961
	11	2·387	9·267		11	1·314	6·873		20	1·308	8·143
	12	2·509	9·868		12	1·380	7·201				
	13	2·620	10·433		13	1·443	7·518	13	14	1·041	6·882
	14	2·728	10·987		14	1·502	7·821		15	1·080	7·086
	15	2·830	11·518		15	1·558	8·113				
									16	1·118	7·285
	16	2·928	12·035		16	1·612	8·398		17	1·153	7·473
	17	3·021	12·532		17	1·664	8·674		18	1·187	7·657
	18	3·110	13·014		18	1·713	8·939		19	1·220	7·837
	19	3·199	13·494		19	1·760	9·197		20	1·251	8·008
	20	3·278	13·936		20	1·806	9·449				
								14	15	1·037	7·007
4	5	1·171	4·783	8	9	1·074	6·037		16	1·073	7·197
					10	1·144	6·368		17	1·108	7·382
	6	1·325	5·417						18	1·140	7·557
	7	1·464	6·018		11	1·208	6·680		19	1·172	7·730
	8	1·593	6·595		12	1·269	6·982		20	1·202	7·896
	9	1·711	7·141		13	1·327	7·274				
	10	1·821	7·663		14	1·381	7·552				
					15	1·433	7·822	15	16	1·035	7·128
	11	1·925	8·077						17	1·067	7·300
	12	2·022	8·648		16	1·483	8·085		18	1·099	7·472
	13	2·113	9·108		17	1·530	8·336		19	1·129	7·637
	14	2·200	9·554		18	1·575	8·580		20	1·159	7·801
	15	2·283	9·985		19	1·619	8·819				
					20	1·661	9·051	16	17	1·032	7·235
	16	2·362	10·402						18	1·062	7·398
	17	2·437	10·803	9	10	1·064	6·244		19	1·092	7·560
	18	2·509	11·192		11	1·125	6·539		20	1·120	7·715
	19	2·578	11·569		12	1·182	6·821				
	20	2·645	11·938		13	1·235	7·090	17	18	1·030	7·337
5	6	1·131	5·192		14	1·286	7·351		19	1·058	7·491
	7	1·250	5·706		15	1·334	7·601		20	1·085	7·640
	8	1·360	6·198								
	9	1·461	6·665		16	1·380	7·844	18	19	1·028	7·432
	10	1·568	7·152		17	1·424	8·079		20	1·054	7·577
					18	1·466	8·306				
	11	1·643	7·539		19	1·507	8·529				
	12	1·726	7·950		20	1·546	8·744	19	20	1·026	7·521

A BIBLIOGRAPHY ON THE THEORY OF QUEUES

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In this bibliography an attempt has been made to assemble all papers on those aspects of the theory of probability which may be grouped together under the heading of the study of queues. In addition to purely theoretical work, the practical applications of the subject have received much attention, the most important and fruitful of these being in the field of telephone traffic. The other main applications have been to the study of road traffic, the allocation of operatives to the servicing of machines, the mathematical aspects of inventory control and production scheduling, storage problems such as the optimal size of dams and miscellaneous topics which include the scheduling of air traffic and the design of appointment systems in hospital outpatient departments. The study of point processes and, in particular, the problems arising from counts obtained from Geiger-Müller counters, resemble closely problems arising in telephone systems where no waiting is possible (loss-systems) and a selection of papers on such processes has been included. Similarly, problems arising in the theory of dams and provisioning are closely related to the problem of collective risk in actuarial studies. In the view of the large number of papers on the latter topic, some of which are specifically actuarial in character, the choice in this case has been limited to a few papers only, these being chosen both for their own intrinsic importance and for the fact that they themselves provide references to other work. A similar policy has been followed for papers concerned strictly with the economics of inventory control and also for papers on renewal theory.

Classification. The papers are listed in alphabetical order by authors and have been classified as belonging to one or more of ten main groups. These are:

C Problems dealing with storage (content).
F Problems relating to flow through a network.
G Applications not covered by the other categories.
I Inventory problems.
M Problems arising in servicing automatic machines.
P Point processes and counter problems.
Q The general theory of queues.
R Road traffic and related topics.
S Stochastic processes directly related to the study of queues.
T Problems in telephone traffic.

Within these groups, a further subdivision has been made into theory (t), numerical results (including tables and graphs) and practical applications (a), and expository and descriptive articles (d). Where it is relevant, a distinction is made between loss-systems (l) in which the customer leaves immediately if service is not available at the instant at which he first requires it and delay-systems (w) in which waiting is possible. The description of a queueing system proposed by D. G. Kendall (*Ann. Math. Statist.* (1953), **24**, 338–54) has been given where possible; in some cases the description has been omitted if the paper in question deals equally with many different systems or if the necessary information for

such a description is lacking. For simplicity, variants of the basic queue discipline (service in order of arrival) are grouped together under the letter v; such variants include queueing with priorities, random choice of the next customer to be served from amongst those awaiting service and queues in series in which the customer receives attention at more than one counter before he leaves the system.

Abstracts and short notes are indicated by the letter A and papers with useful lists of references by the letter B. Finally, the papers judged to be the most important either by reason of their contents or for their survey of a branch of the subject have been marked with an asterisk.

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MISCELLANEA

Studies in the history of probability and statistics
VI. A note on the early solutions of the problem of the duration of play

By A. R. THATCHER

It is now just 300 years since the publication by Huygens of the first result on the famous problem which became known as the Duration of Play. The aim of this note is to summarize the early development of this problem and to show how easily some of the solutions found at the beginning of the eighteenth century can be linked with modern work on sequential tests, random walks and certain storage problems.

We use throughout the following notation. Call the two players A and B, and let their chances of winning a game be p and $q = 1-p$, respectively. A starts with a counters and B starts with b counters, and after each game the loser hands one counter to the winner. It is desired to find first the probability P_a that A will eventually lose all his counters without having previously won all B's, and more generally the probability $P_{a,n}$ that this will happen within n games. P_b and $P_{b,n}$ are defined similarly. $P_{a,n}+P_{b,n}$ is the probability that the play will terminate (with the 'ruin' of one of the players) within n games. It can be shown that the play must end sooner or later, so that $P_a+P_b = 1$.

In 1657 Huygens gave without proof, in the fifth and last problem of his treatise *De ratiociniis in ludo aleae*, the numerical value for P_a in a case where $a = b = 12$ and where p and q had particular values. The general result for P_a was found by James Bernoulli, who died in 1705, but it remained in manuscript until it was published 8 years later in his *Ars Conjectandi*; Bernoulli says that the proof is laborious and leaves it to the reader. Before the *Ars Conjectandi* appeared, however, de Moivre had found a simple derivation independently and published it in his treatise *De Mensura Sortis* (1711).

De Moivre's original proof, which was later reproduced in his *Doctrine of Chances* (see 1711, pp. 227–8; 1718, pp. 23–4; 1738, pp. 45–6; 1756, pp. 52–3), is very ingenious and so much shorter than the demonstrations usually given in modern textbooks that it is worth quoting. Its essence is as follows. Imagine that each player starts with his counters before him in a pile, and that nominal values are assigned to the counters in the following manner. A's bottom counter is given the nominal value q/p; the next is given the nominal value $(q/p)^2$, and so on until his top counter which has the nominal value $(q/p)^a$. B's top counter is valued $(q/p)^{a+1}$, and so on downwards until his bottom counter which is valued $(q/p)^{a+b}$. After each game the loser's top counter is transferred to the top of the winner's pile, and it is always the top counter which is staked for the next game. Then *in terms of the nominal values* B's stake is always q/p times A's, so that at every game each player's nominal expectation is nil. This remains true throughout the play; therefore A's chance of winning all B's counters, multiplied by his nominal gain if he does so, must equal B's chance multiplied by B's nominal gain. Thus

$$P_b\left\{\left(\frac{q}{p}\right)^{a+1}+\left(\frac{q}{p}\right)^{a+2}+\ldots+\left(\frac{q}{p}\right)^{a+b}\right\} = P_a\left\{\left(\frac{q}{p}\right)+\left(\frac{q}{p}\right)^{2}+\ldots+\left(\frac{q}{p}\right)^{a}\right\}.$$

The use of $P_a+P_b = 1$ now gives immediately

$$P_b = \frac{(q/p)^a-1}{(q/p)^{a+b}-1}, \tag{1}$$

and this is the probability of the 'gambler's ruin'.

In terms of the counters, A's total expected gain is bP_b-aP_a, while his expectation per game is $p-q$. These obvious facts are indeed only special cases of a more general result given by de Moivre (1718, pp. 135–6; 1738, pp. 48–9; 1756, pp. 55–6). De Moivre does not actually divide one expression by the other, but, since the total expectation equals the expectation per game times the expected number of games, this division is all that is required in order to get the expected number of games

$$E(N) = \frac{bP_b-aP_a}{p-q}. \tag{2}$$

De Moivre was also the first to discover and publish a general method for calculating $P_{a,n}+P_{b,n}$, thus finding the chance that the play would terminate within n games. For the case where a is infinite (so that $P_{a,n} = 0$) and $n-b$ is odd, he found

$$P_{b,n} = \text{first } \tfrac{1}{2}(n-b+1) \text{ terms of } (p+q)^n + \text{first } \tfrac{1}{2}(n-b+1) \text{ terms of } (p/q)^b(q+p)^n. \tag{3}$$

This solution, with a similar one for the case where $n-b$ is even, was given without proof in his *De Mensura Sortis* and later in *The Doctrine of Chances* (1711, p. 262; 1718, pp. 119–20; 1738, p. 179; 1756, pp. 208–9). Fieller (1931) has drawn attention to this result and also provided a simple and elegant proof.

De Moivre's first solution of the general problem of calculating $P_{a,n}+P_{b,n}$ when both a and b are finite (1711, p. 261; 1718, pp. 113–14; 1738, stated incorrectly on pp. 173–4; 1756, p. 203) called for the performance of $n-1$ multiplications and the rejection of certain terms during the process. For moderate n the calculation is not so tedious as appears at first sight, and it has the advantage of giving the answer reduced to the smallest number of terms; as de Moivre later pointed out, the rejected terms can also be used to obtain $P_{a,n}$ and $P_{b,n}$ separately.

However, a few months before de Moivre's method actually appeared (for the *Philosophical Transactions* for 1711 were delayed in the press), a different solution giving $P_{a,n}$ and $P_{b,n}$ separately had been found and was soon published by de Montmort (1713). This result is of particular interest because it provides one of the easiest solutions of the problem, since the series which can be derived from it by using modern tables is rapidly convergent over the range of values of n where the play is likely to terminate.

In 1710 de Montmort found a method for calculating $P_{a,n}$ and $P_{b,n}$ for the case $p=q$. He sent some numerical results to John Bernoulli, who passed the letter to his nephew Nicholas. In a reply dated 26 February 1711, published by de Montmort (1713, p. 308 et seq.), Nicholas Bernoulli gave without proof the general solution for the case $p \neq q$; in modern notation it can be written as follows:

$$P_{b,n} = \sum_t \left\{ p^{ts+b} q^{ts} \sum_i \binom{n}{i} (p^{n-b-2ts-i} q^i + q^{n-b-2ts-i} p^i) \right\} - \sum_t \left\{ p^{ts+s} q^{ts+a} \sum_i \binom{n}{i} (p^{n-b-2ts-2a-i} q^i + q^{n-b-2ts-2a-i} p^i) \right\}. \tag{4}$$

In this formula $s=a+b$; the summation over $i \geqslant 0$ continues until the terms in the series in each curly bracket, re-arranged in descending powers of p, meet in the middle (the middle term counting only once if $n-b$ is even); and the summation over t covers all values $\geqslant 0$ which leave non-negative exponents within the summation over i on the line concerned. Bernoulli stated the result for $n-b$ even, but in fact (4) is also valid if $n-b$ is odd.

Not content with this, Nicholas Bernoulli confirmed that the limit of (4) as $n \to \infty$ gives the correct value for P_b. He does not give his method but it is not difficult to guess; if for example $p>q$ it is only necessary to re-write the two lines of (4) as

$$\sum_t p^{-ts} q^{ts} [p^n + np^{n-1}q + \ldots + p^{2ts+b} q^{n-2st-b}] - \sum_t p^{-ts-a} q^{ts+a} [p^n + np^{n-1}q + \ldots + p^{2ts+2a+b} q^{n-2ts-2a-b}]. \tag{5}$$

As $n \to \infty$ the sums in each square bracket tend to 1; this follows from (James) Bernoulli's Theorem, which at the time had not been published but which was known to Nicholas. The expression thus reduces to two geometric series, and is immediately seen to agree with (1) above. In passing, it may be noted that as $a \to \infty$ the expression (4) reduces to de Moivre's expression (3).

When de Montmort saw this extraordinary solution he admitted that he could not follow it (this was partly because Bernoulli had inadvertently used one symbol in two senses), and remarked: 'votre formule m'étonne pour sa generalité' (1713, p. 316). Later, in comparing it with his own, he said: 'je n'ai eu en vûe que la supposition des hazards égaux pour l'un et pour l'autre Joueur, au lieu que vous les supposés dans un rapport quelconque' (1713, p. 345). De Montmort's solution, which he then describes briefly, consisted of a method of picking out the binomial coefficients in (4) from Pascal's triangle; this was of course sufficient when $p=q$, and was in itself a remarkable result to have found. Nevertheless, it seems clear that the solution (4) of the general case $p \neq q$, though often described as de Montmort's, was in fact found first by Nicholas Bernoulli.

De Montmort reproduced (4) in the body of his book, gave an example and added a most interesting though far from rigorous demonstration (1713, pp. 268–72). De Moivre at first called the result 'very handsom' (1718, p. 122), but later criticized de Montmort's statement of it (which indeed is not entirely correct) and seems to hint that he had found the same method of solution before the year 1711 (see 1738, pp. 181–2; 1756, pp. 210–11). This is certainly possible, though it may be doubted whether de Moivre had carried the investigation of (4) as far as Bernoulli; perhaps he used it in particular cases, but did not pursue the matter because his own result gave $P_{a,n}+P_{b,n}$ in a smaller number of terms.

De Moivre later solved the Duration of Play problem in two further ways, and in the course of his work made an extensive investigation of recurring series (which he was the first to explore). His results included the partial fraction expansion of a generating function (1738, pp. 197–99; 1756, pp. 224–7); he found the

probability of runs of successes (1738, pp. 243–8; 1756, pp. 254–9), and of course made the original derivation of the normal distribution (1738, pp. 235–43; 1756, pp. 243–50). On the Duration of Play problem itself he expressed $P_{b,n}$ as a recurring series with fewer terms than (4); and finally he discovered the first results on the trigonometrical solution (see Feller, 1950, p. 292, equation 5·7), including the asymptotic form for $P_{b,n}$ when $a = b$ and $p = q$. For fuller details of his work, and of its subsequent development by Laplace and many others, the reader is referred to Todhunter (1865) and Fieller (1931).

It remains to show the link between these early solutions and modern work. This stems from the well-known fact that the Duration of Play situation can be regarded as a linear random walk with two absorbing barriers, such that the movement of the particle at each jump has a distribution with mean $\mu = p - q$ and variance $\sigma^2 = 4pq$. To complete the comparison a simple approximation is required, namely

$$(p/q)^\lambda \doteq \exp(2\lambda\mu/\sigma^2), \tag{6}$$

which can be shown to apply with sufficient accuracy in the cases for which it will be required.

If then in equations (1) and (2) we make the substitutions (6) and $p - q = \mu$ we shall obtain approximations for the probability of absorption at a given barrier, and for the expected number of steps before absorption at either barrier, in the corresponding random walk; and under the conditions of the central limit theorem these will be valid for all walks with given finite μ and σ, provided that the number of steps is sufficiently large. It can be seen by inspection that the transformed version of equations (1) and (2) are in fact the same as Wald's approximations for the operating characteristic and average sample number of a sequential test, in the form quoted by Page (1954, equations 5, 7).

We can similarly transform (3), making the normal approximation to the binomial expressions; it will be found that the result agrees with that given by Bartlett (1946, equation 8), obtained as the solution of a differential equation for the diffusion process. It is of interest to note that the same result can also be used to find a quick approximate solution of a storage problem considered in a recent paper by Anis (1956). This concerns a reservoir, of unlimited capacity, which has initial water level x; this level varies each year by an amount distributed with zero mean and unit variance. When n and x are sufficiently large we can ignore the end-effects and assume that the probability that the reservoir will run dry within n years is approximately the same as the probability that B will lose $b = x$ counters within n trials (where a is infinite and $p = q = \frac{1}{2}$). By de Moivre's result (3) this probability will be twice the sum of the first $\frac{1}{2}(n - x + 1)$ terms of $(\frac{1}{2} + \frac{1}{2})^n$. Hence, for large n and x the probability that the reservoir will not run dry within n years can be expressed approximately as $2\int_0^{x/\sqrt{n}} e^{-\frac{1}{2}t^2}/\sqrt{(2\pi)}\, dt$, and it is easy to verify that this distribution has the same moment ratios as the limiting values found by Anis.

Finally, we come to Nicholas Bernoulli's general solution of the Duration of Play. If for any value of t either line of (4) is arranged in descending powers of p, it will be found to be the sum of multiples of two binomial expressions in the same way as (3)—see also Fieller (1931, equation 10.1), who proceeds to obtain the exact solution of the problem in a convenient form as a series of multiples of incomplete beta-functions, and also provides a rigorous proof.

The application of (6) and the normal approximation to the binomial puts the solution in the simple approximate form $\Sigma A_i \int_{a_i}^{\infty} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}x^2}\, dx$; this series agrees with the (exact) result given by Bartlett (1946, equation 17) for the diffusion process. In view of the usefulness of this series it is worth repeating here for completeness

$$P_{b,n} \doteq F(b) - w(-a)\,F(b+2a) + w(-a-b)\,F(3b+2a) - w(-2a-b)\,F(3b+4a) + \ldots, \tag{7}$$

where

$$F(\lambda) \equiv Q\left(\frac{\lambda}{\sigma\sqrt{n}} - \frac{\mu\sqrt{n}}{\sigma}\right) + w(\lambda)\,Q\left(\frac{\lambda}{\sigma\sqrt{n}} + \frac{\mu\sqrt{n}}{\sigma}\right),$$

$$Q(\lambda) \equiv \int_{\lambda}^{\infty} \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}x^2}\, dx,$$

$$w(\lambda) \equiv \exp(2\lambda\mu/\sigma^2).$$

The corresponding series for $P_{a,n}$ is found by interchanging a with b and changing the sign of μ in the definitions of F and w.

It will be found that (7) converges rapidly over the range of n where the process is likely to terminate, and so (as suggested by Bartlett) provides a rapid approximation for the probability that a particle starting at the origin, with a jump distribution having mean μ and variance σ^2, will reach $x = b$ (without having previously been absorbed at $x = -a$) within n jumps. It can similarly be used to find the chance that a linear sequential test will end within n trials, or that a finite reservoir with random net input will either dry up or overflow within a given time.

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Optimal sampling for quota fulfilment

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1. The problem to be discussed in this paper arises in the following way. It is desired to obtain a sample from a stratified population in such a way that there are exactly m_i individuals from stratum $\omega_i (i = 1, \ldots, k)$. It is more convenient to take a random sample from the whole population, and to ascertain subsequently the strata to which the chosen individuals belong, than to search for individuals belonging to specified strata. Therefore, a first sample of N individuals is chosen without regard to stratification and any shortfall is made up by a further set of samples, each restricted to one of the deficient strata, and of such a size as to provide the required number of individuals from each of the strata. Thus, if the first sample of N contains $n_i (< m_i)$ individuals from stratum ω_i then the subsequent sample from this stratum will contain $m_i - n_i$ individuals but if $n_i \geqslant m_i$, no subsequent sample from this stratum will be required.

If c is the cost per individual in the first (unrestricted) sample, and c_i the cost per individual for a sample restricted to stratum ω_i, then the expected cost of obtaining the complete sample is

$$C_1 = cN + \sum_{i=1}^{k} c_i \mathscr{E}(m_i - n_i \,|\, n_i < m_i) \Pr\{n_i < m_i\}, \tag{1}$$

where n_i is the number of observations included in ω_i in the first unrestricted sample. If the unused individuals in stratum ω_i with numbers in excess of requirements are worth c_i' each the expected cost is

$$C_2 = cN + \sum_{i=1}^{k} [c_i \mathscr{E}(m_i - n_i \,|\, n_i < m_i) \Pr\{n_i < m_i\} + c_i' \mathscr{E}(m_i - n_i \,|\, n_i \geqslant m_i) \Pr\{n_i \geqslant m_i\}]. \tag{2}$$

C_1 can, of course, be regarded as a special case of C_2.

2. If it is supposed that the joint distribution of $n_1, n_2, \ldots, n_k$ is multinomial with parameters $p_1, p_2, \ldots, p_k$ (as would be appropriate if sampling from a large population with proportions $p_1, p_2, \ldots, p_k$ in strata $\omega_1, \omega_2, \ldots, \omega_k$, respectively, were being considered) then

$$\mathscr{E}(m_i - n_i) = m_i - Np_i$$

and (2) can be written

$$C_2 = cN + \sum_{i=1}^{k} (c_i - c_i') \mathscr{E}(m_i - n_i \,|\, n_i < m_i) \Pr\{n_i < m_i\} + \sum_{i=1}^{k} c_i'(m_i - Np_i). \tag{2a}$$

Using Gruder's formula $\displaystyle\sum_{r=0}^{m-1} (r - Np) \binom{N}{r} p^r q^{N-r} = -m \binom{N}{m} p^m q^{N-m+1}$,

C_2 can be expressed in the form

$$C_2 = cN + \sum_{i=1}^{k} (c_i + c_i') \left[(m_i - Np_i) \sum_{j=0}^{m_i - 1} \binom{N}{j} p_i^j q_i^{N-j} + m_i \binom{N}{m_i} p_i^{m_i} q_i^{N - m_i + 1} \right] + \sum_{i=1}^{k} c_i'(m_i - Np_i). \tag{3}$$

If N is large enough and $|\, m_i - Np_i \,|\, (Np_iq_i)^{-\frac{1}{2}}$ is not too large, approximate expressions for the expected cost can be obtained by putting

$$\mathscr{E}(m_i - n_i \mid n_i < m_i)\Pr\{n_i < m_i\} \doteqdot (Np_iq_i)^{\frac{1}{2}}(X_i I(X_i) + Z(X_i)),$$

where
$$X_i = \frac{m_i - Np_i}{(Np_iq_i)^{\frac{1}{2}}}; \quad Z(X_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X_i}; \quad I(X_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{X_i} e^{-\frac{1}{2}t^2} dt.$$

Then
$$C_2 \doteqdot cN + \sum_{i=1}^{k} (c_i - c_i')(Np_iq_i)^{\frac{1}{2}}(X_i I(X_i) + Z(X_i)) + \sum_{i=1}^{k} c_i'(m_i - Np_i). \tag{4}$$

Approximate values of N minimizing C_2 can be found by equating the differential coefficients of (4), with regard to N, to zero. The resulting equation is

$$\tfrac{1}{2}(c_i - c_i')\left(\frac{q_i}{Np_i}\right)^{\frac{1}{2}} Z(X_i) = c_i I(X_i) + c_i'(1 - I(X_i)). \tag{5}$$

3. However, an exact approach leads to a simpler solution in the present instance. If N is increased by unity, the probability that the extra observation comes from ω_i is p_i. The cost of obtaining the extra observation is c, but there is an amount c_i to set against this if ω_i is a deficient stratum or c_i' if the first sample contains m_i or more individuals from ω_i. Hence, the change in the expected cost is

$$\Delta C_2 = c - \sum_{i=1}^{k} p_i[(c_i - c_i')\Pr\{n_i < m_i\} + c_i']. \tag{6}$$

Optimal values of N are obtained as the least value of N for which $\Delta C_2 > 0$. These optimal values will approximately satisfy the equation $\Delta C_2 = 0$, i.e.

$$\sum_{i=1}^{k} \frac{(c_i - c_i')p_i}{c - \sum_{j=1}^{k} c_j' p_j} \Pr\{n_i < m_i\} = 1, \tag{7}$$

or
$$\sum_{i=1}^{k} g_i \Pr\{n_i < m_i\} = 1, \tag{7a}$$

where
$$g_i = \frac{(c_i/c) - (c_i'/c)}{1 - \sum_{j=1}^{k} (c_j'/c) p_j} p_i.$$

If
$$c_i' = 0; \quad g_i = (c_i/c)p_i.$$

In the special case of complete symmetry where

$$p_i = 1/k; \quad c_i/c = d; \quad c_i'/c = d'; \quad m_i = m,$$

then $g_i = \dfrac{1}{k}\dfrac{d - d'}{1 - d'}$ and (7a) becomes

$$\Pr\{n_i < m\} = \frac{1}{kg} = \frac{1 - d'}{d - d'}. \tag{8}$$

4. Table 1 gives optimal values of N for this symmetrical case for

$$k = 2(1)10,$$

$$m = \text{nearest integer to } 50k^{-1},\ 100k^{-1},\ 200k^{-1},\ 500k^{-1},$$

$$c_i/c = d = 1{\cdot}25,\ 1{\cdot}5,\ 2{\cdot}0,\ 2{\cdot}5,\ 3{\cdot}0,$$

$$c_i'/c = d' = 0{\cdot}9,\ 0{\cdot}7,\ 0{\cdot}25,\ 0.$$

Values of c_i/c greater than one and of c_i'/c less than one, were chosen for the following reasons. If *any* c_i's are less than (or equal to) c it is best to obtain the required sample of m_i from those strata for which $c_i \leqslant c$ by restricted sampling, and then to use the optimal N for the remaining strata. Similarly, if *all* the c_i''s are greater than c the expected cost decreases without limit as N increases.

The values of N in Table 1 were obtained by using the tables of the incomplete beta-function (ed. K. Pearson (1934)) so far as the range of argument of these tables extended. This method could be used for nearly all combinations of d and d' when $km = 50$ and for some values of d and d' when $km = 100$ and

$k = 2$. For higher values of km (the total size of the required sample) calculations were based on the approximate formula

$$\frac{k(m-\frac{1}{2})-N}{(N(k-1))^{\frac{1}{2}}} \doteqdot \lambda_{1/k\varepsilon}, \tag{9}$$

where λ is defined by

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda_\alpha} e^{-\frac{1}{2}t^2}dt = \alpha.$$

Equation (9) leads to

$$N \doteqdot k(m-\tfrac{1}{2})\,[(1+y^2)^{\frac{1}{2}}-y]^2, \tag{10}$$

where

$$y = \tfrac{1}{2}\lambda_{1/k\varepsilon}\left(\frac{k-1}{k(m-\frac{1}{2})}\right)^{\frac{1}{2}}.$$

As the size, km, of the required sample increases, y tends to zero and

$$N \sim k(m-\tfrac{1}{2})\,(1-2y+2y^2) = k(m-\tfrac{1}{2})-\lambda_{1/k\varepsilon}(k(k-1)\,(m-\tfrac{1}{2}))^{\frac{1}{2}}+\tfrac{1}{2}\lambda^2_{1/k\varepsilon}(k-1). \tag{11}$$

This formula gives values for N which are not in error by more than one when $km \doteqdot 50$ and can be used with reasonable confidence when $km \geqslant 100$ provided k is not too large.

Table 1 shows that the optimal size N for the first sample increases with d and with d', as might be expected on intuitive grounds. For $k = 2$ the optimum N differs only slightly from the required sample size km, but as k increases the variation in optimal N with d and d' becomes more pronounced.

The minimized value of C_2 is

$$C_{2(\min)} = c[N + k(d-d')\,\mathscr{E}(m-n_i \mid n_i < m)\,\Pr\{n_i < m\} + kd'(m-(N/k))].$$

Since N is chosen to satisfy approximately the equation

$$\Pr\{n_i < m\} = (1-d')/(d-d'), \tag{8 bis}$$

$$C_{2(\min)} \doteqdot c[N + k(1-d')\,\{m-(N/k)-\mathscr{E}(n_i-(N/k) \mid n_i < m)\} + kd'(m-(N/k))],$$

i.e.

$$C_{2(\min)} \doteqdot cm\left[k + (d-d')\binom{N}{m}(k-1)^{N-m+1}k^{-N}\right],$$

so that the ratio of the minimized cost to the cost of choosing the whole sample by restricted sampling within each stratum is

$$\frac{C_{2(\min)}}{mckd} \doteqdot \frac{1}{d} + \left(1-\frac{d'}{d}\right)\left(1-\frac{1}{k}\right)^{N+1}\binom{N}{m}(k-1)^{-m}. \tag{12}$$

Values of this ratio are given in Table 2 (calculated by Miss E. J. Smith) for the following values of k, m, d and d'

$k = 2, 3, 4, 5,$ and $10,$

$m =$ nearest integer to $50k^{-1}$, $100k^{-1}$, $500k^{-1}$,

$d = 1{\cdot}5,\ 2{\cdot}5,\ 3{\cdot}0,$

$d' = 0{\cdot}5,\ 0{\cdot}1,\ 0{\cdot}0.$

It may be noted that when $N < km$ some restricted samples will be needed, and even for the optimal values of N in Table 1 the probability of requiring additional samples must be quite high.

REFERENCE

K. Pearson (ed.) (1934). *Tables of the Incomplete Beta-Function.* Cambridge University Press.

Table 1

k	Required sample size (km)	d / d'	1·25	1·5	2·0	2·5	3·0	k	Required sample size (km)	d / d'	1·25	1·5	2·0	2·5	3·0
2	50	0·9	53	56	59	61	62	5	50	0·9	56	62	67	70	72
		0·7	48	51	54	56	58			0·7	47	53	59	62	64
		0·25	44	47	50	52	53			0·25	39	45	51	54	57
		0	43	46	49	51	52			0	37	42	48	52	54
	100	0·9	106	109	113	116	117		100	0·9	109	118	127	132	136
		0·7	98	102	107	110	111			0·7	95	104	113	118	123
		0·25	93	97	101	103	105			0·25	85	93	101	106	110
		0	91	95	99	102	103			0	82	89	97	103	106
	200	0·9	207	213	219	222	224		200	0·9	214	227	235	245	250
		0·7	198	204	210	213	216			0·7	195	207	219	227	232
		0·25	190	196	202	205	208			0·25	181	191	203	210	215
		0	187	193	199	203	205			0	177	186	197	205	210
	500	0·9	512	521	530	534	538		500	0·9	523	543	561	571	578
		0·7	497	506	516	521	525			0·7	493	512	531	543	551
		0·25	483	493	503	509	513			0·25	468	486	506	517	524
		0	481	489	499	505	509			0	461	479	497	509	517
3	51	0·9	55	60	65	67	69	6	48	0·9	55	61	68	73	77
		0·7	49	53	57	60	62			0·7	44	51	58	61	63
		0·25	43	47	51	54	56			0·25	36	42	49	53	55
		0	42	45	49	52	54			0	34	39	46	50	53
	99	0·9	106	111	118	121	124		102	0·9	112	123	133	139	143
		0·7	96	102	108	111	115			0·7	97	106	117	123	127
		0·25	89	94	100	104	108			0·25	85	94	103	109	113
		0	87	92	98	101	104			0	82	90	99	105	109
	201	0·9	213	222	229	234	237		198	0·9	213	228	241	249	254
		0·7	199	208	216	222	227			0·7	192	205	219	228	234
		0·25	188	196	205	210	213			0·25	175	187	201	209	215
		0	185	193	201	205	208			0	170	182	195	203	209
	501	0·9	518	532	543	550	555		498	0·9	524	545	566	577	585
		0·7	496	510	524	532	537			0·7	490	511	533	545	555
		0·25	479	492	505	513	519			0·25	463	483	504	517	526
		0	474	486	499	508	513			0	455	474	495	508	517
4	52	0·9	58	63	67	69	71	7	49	0·9	57	64	73	77	81
		0·7	49	54	60	63	65			0·7	45	52	60	63	67
		0·25	43	47	53	56	58			0·25	36	43	50	54	58
		0	41	45	50	54	56			0	34	40	46	51	54
	100	0·9	108	116	123	128	131		98	0·9	109	120	132	138	143
		0·7	96	104	111	116	119			0·7	92	102	114	120	125
		0·25	87	94	101	106	109			0·25	80	89	99	105	110
		0	85	91	98	103	106			0	77	85	94	101	105
	200	0·9	212	223	233	239	243		203	0·9	221	237	253	262	268
		0·7	196	205	217	223	228			0·7	196	211	228	237	244
		0·25	182	192	203	209	213			0·25	177	191	206	216	222
		0	179	188	198	204	209			0	171	185	199	209	216
	500	0·9	520	537	552	561	567		497	0·9	531	555	578	590	599
		0·7	494	510	527	537	544			0·7	494	517	541	555	566
		0·25	473	488	505	515	522			0·25	464	486	509	524	534
		0	467	482	498	508	515			0	456	477	499	514	524

Table 1 (*cont.*)

k	Required sample size (km)	d / d'	1·25	1·5	2·0	2·5	3·0
8	48	0·9	56	64	73	79	83
		0·7	43	51	59	64	68
		0·25	34	40	48	53	56
		0	31	38	45	50	53
	104	0·9	116	129	142	149	154
		0·7	97	109	121	129	134
		0·25	84	94	105	112	117
		0	80	89	100	107	111
	200	0·9	228	235	252	261	268
		0·7	192	208	225	235	243
		0·25	173	187	203	213	220
		0	167	178	196	206	213
	504	0·9	533	561	585	599	608
		0·7	494	519	545	561	572
		0·25	462	485	511	526	537
		0	453	475	500	515	526
9	54	0·9	62	73	83	89	94
		0·7	49	56	66	73	77
		0·25	38	46	55	59	63
		0	36	43	51	56	59
	99	0·9	111	125	138	146	152
		0·7	92	104	117	125	130
		0·25	77	88	100	107	113
		0	74	84	94	102	107
9	198	0·9	217	235	253	263	270
		0·7	189	206	225	235	243
		0·25	169	184	201	211	219
		0	163	177	193	204	211
	504	0·9	537	565	591	606	616
		0·7	493	520	548	565	577
		0·25	459	484	511	528	539
		0	449	473	499	516	528
10	50	0·9	59	69	80	86	91
		0·7	44	53	62	69	73
		0·25	34	41	50	55	59
		0	31	38	46	52	55
	100	0·9	113	127	142	150	156
		0·7	92	105	119	127	134
		0·25	73	88	101	108	114
		0	67	83	95	103	108
	200	0·9	220	240	259	270	277
		0·7	191	208	228	240	248
		0·25	169	185	203	214	222
		0	163	178	195	205	214
	500	0·9	534	564	592	608	619
		0·7	489	517	547	564	577
		0·25	451	478	508	525	537
		0	442	467	495	512	525

Table 2. *Ratio of expected cost of optimal sample to cost of restricted sampling*

k	m	d'	d		
			1·5	2·5	3·0
2	25	0·5	0·704	0·437	0·369
		0·1	0·711	0·451	0·382
		0·0	0·716	0·455	0·386
2	50	0·5	0·693	0·426	0·357
		0·1	0·700	0·437	0·368
		0·0	0·703	0·439	0·371
2	250	0·5	0·679	0·412	0·344
		0·1	0·682	0·416	0·348
		0·0	0·683	0·417	0·350
3	17	0·5	0·719	0·452	0·383
		0·1	0·733	0·473	0·403
		0·0	0·733	0·478	0·408
3	33	0·5	0·704	0·438	0·368
		0·1	0·717	0·453	0·382
		0·0	0·717	0·456	0·386
3	167	0·5	0·682	0·416	0·348
		0·1	0·688	0·423	0·355
		0·0	0·689	0·425	0·357
4	13	0·5	0·730	0·465	0·386
		0·1	0·751	0·490	0·417
		0·0	0·749	0·493	0·423
4	25	0·5	0·712	0·446	0·377
		0·1	0·728	0·464	0·394
		0·0	0·727	0·468	0·398
4	125	0·5	0·687	0·420	0·352
		0·1	0·694	0·428	0·360
		0·0	0·694	0·430	0·362
5	10	0·5	0·741	0·476	0·404
		0·1	0·760	0·504	0·434
		0·0	0·762	0·510	0·440
5	20	0·5	0·719	0·455	0·382
		0·1	0·732	0·474	0·404
		0·0	0·735	0·479	0·409
5	100	0·5	0·690	0·424	0·355
		0·1	0·697	0·433	0·365
		0·0	0·699	0·435	0·366
10	5	0·5	0·786	0·515	0·445
		0·1	0·801	0·558	0·487
		0·0	0·806	0·565	0·493
10	10	0·5	0·745	0·482	0·409
		0·1	0·763	0·512	0·441
		0·0	0·767	0·518	0·448
10	50	0·5	0·702	0·442	0·366
		0·1	0·711	0·450	0·380
		0·0	0·714	0·453	0·383

The distribution of intervals between successive maxima in a series of random numbers

By D. S. PALMER

Marconi's Wireless Telegraph Co., Ltd

Let $u(r)$ be the rth in order of occurrence of a series of independent random numbers. The $n+3$ numbers $u(-1), u(0), \ldots, u(n+1)$ can occur in $(n+3)!$ different orders when placed in ascending order of magnitude, and these orders have all equal probability independently of the frequency distribution of the u's; cases where two or more u's are equal are here considered to have probability zero.

Let $Q(n)$ be the number of these orders which have $u(-1) < u(0) > u(1)$, but no further maxima, although $u(n+1)$ may exceed $u(n)$. Of these, let $q(m,n)$ have $u(n+1)$ in position m from the bottom. Then

$$Q(n) = \sum_{m=1}^{n+3} q(m,n). \tag{1}$$

Consider the $Q(n)$ orders, and the positions in which a further number $u(n+2)$ may be placed to make an order of the group $Q(n+1)$. Four cases must be distinguished

(A) $$u(n+1) < u(n), \quad u(n+1) < u(-1).$$

$u(n+1)$ must have the lowest place and $u(-1)$ may have any of $n+1$ places in the ordering of $u(-1) \ldots u(n+1)$. Hence $q(1,n) = n+1$. $u(n+2)$ may have any place, so this case contributes $n+1$ to each of $q(1,n+1), q(2,n+1) \ldots q(n+4,n+1)$.

(B) $$u(n+1) < u(n), \quad u(n+1) > u(-1).$$

Only one ordering of $u(-1) \ldots u(n+1)$ is possible, and $u(n+2)$ may have any place. There arises a contribution of 1 to $q(1,n+1), q(2,n+1), \ldots, q(n+4,n+1)$.

(C) $u(n+1) > u(n)$ and $u(n+1)$ has second place.

There are n possible positions of $u(-1)$ with regard to $u(0) \ldots u(n+1)$. $u(n+2)$ may have any place but the two lowest. There arises a contribution of n to $q(m,n+1)$ for each $m > 2$.

(D) $u(-1) \ldots u(n+1)$ belong to $q(r,n)$ $(r > 2)$.

$u(n+2)$ must $> u(n+1)$, contributing $q(r,n)$ to each $q(m,n+1)$ with $m > r$.

From cases (A), (B), (C) it follows that

$$q(1,n) = q(2,n) = n+1. \tag{2}$$

Summarizing the contributions to $q(m,n+1)$ we have

	$q(1,n+1)$	$q(2,n+1)$	$q(m,n+1)$
(A)	$n+1$	$n+1$	$n+1$
(B)	1	1	1
(C)	0	0	n $(m>2)$
(D)	0	0	$q(r,n)$ $(m>r)$

Therefore $q(1,n+1) = q(2,n+1) = n+2$, agreeing with (2) and

$$q(m,n+1) = 2(n+1) + \sum_{r=3}^{m-1} q(r,n). \tag{3}$$

Assume that

$$q(m,n) = 2^{m-2}\left(n - \frac{m-3}{2}\right), \quad 3 \leqslant m \leqslant n+3. \tag{4}$$

Then
$$q(m,n+1) = 2(n+1) + \sum_{r=3}^{m-1} 2^{r-2}\left(n - \frac{r-3}{2}\right)$$
$$= 2(n+1) + \left(n+\frac{3}{2}\right)\sum_{r=3}^{m-1} 2^{r-2} - \frac{1}{2}\sum_{r=3}^{m-1} r2^{r-2}$$
$$= 2(n+1) + (2n+3)\sum_{r=0}^{m-4} 2^r - \sum_{r=0}^{m-4}(r+3)\,2^r$$
$$= 2(n+1) + 2n\sum_{r=0}^{m-4} 2^r - \sum_{r=0}^{m-4} r2^r$$
$$= 2(n+1) + 2n(2^{m-3}-1) - (m-5)\,2^{m-3} - 2$$
$$= 2^{m-2}\left(n+1-\frac{m-3}{2}\right) \quad \text{agreeing with (4).}$$

(4) is valid for $n = 1$ and 2 as may be seen by the following enumeration. Orders for which $u(n+1) < u(n)$ are enclosed in brackets.

$n = 1$ $q(1,1) = 2$ (2431), (3421),
$q(2,1) = 2$ (1432), 3412,
$q(3,1) = 2$ 2413, 1423,
$q(4,1) = 2$ 1324, 2314.

$n = 2$ $q(1,2) = 3$ (25431), (45321), (35421),
$q(2,2) = 3$ (15432), 45312, 35412,
$q(3,2) = 4$ 15423, 25413, 45123, 45213,
$q(4,2) = 6$ 15234, 15324, 25134, 25314, 35124, 35214,
$q(5,2) = 8$ 13245, 14325, 14235, 23145, 24135, 24315, 34125, 34215.

Thus the validity of (4) is proved by induction.

By (3)
$$q(n+4,n+1) = 2(n+1) + \sum_{r=3}^{n+3} q(r,n)$$
$$= \sum_{r=1}^{n+3} q(r,n) \text{ by (2)}$$
$$= Q(n) \text{ by (1).}$$

Hence, by (4),
$$Q(n) = 2^{n+2}\{n+1-\tfrac{1}{2}(n+1)\} = 2^{n+1}(n+1). \tag{5}$$

Of the $(n+3)!$ orders of $u(-1)\ldots u(n+1)$, one-third satisfy the condition that $u(-1) < u(0) > u(1)$. Therefore

$P(n)$ = chance of an interval of n between successive maxima
$$= Q(n-1)/\tfrac{1}{3}(n+2)! - Q(n)/\tfrac{1}{3}(n+3)!$$
$$= \frac{3n\,2^n}{(n+2)!} - \frac{3(n+1)\,2^{n+1}}{(n+3)!} = 3\cdot 2^n(n-1)\,(n+2)/(n+3)! \quad (n \geqslant 2). \tag{6}$$

Then for the moments we have
$$E(e^{nt}) = 3\sum_{n=2}^{\infty} e^{nt}2^n(n-1)\,(n+2)/(n+3)!$$
$$= e^{2e^t}\{\tfrac{3}{2}e^{-t} - 3e^{-2t} + \tfrac{3}{2}e^{-3t}\} - \tfrac{3}{2}e^{-3t} + \tfrac{3}{2}e^{-t} + 1$$
$$= 1 + 3t + (\tfrac{3}{2}e^2 - 6)\,t^2 + \ldots. \tag{7}$$

The first two terms are obviously correct, and the third gives
$$\text{S.D.} = (3e^2 - 21)^{\frac{1}{2}} = 1{\cdot}080.$$

The first 2000 two-figure random numbers in Kendall and Babington Smith's Tables (*Tracts for Computers No. XXIV*) give

n	$P(n)$	Expected	Observed	χ^2
2	2/5	266·0	251	0·85
3	1/3	221·7	241	1·71
4	6/35	114·0	107	0·43
5	1/15	44·3	50	0·73
$\geqslant 6$	1/35	19·0	16	0·47
		665	665	

Mean 3·01, S.D. 1·04, χ^2 (4 d.f.) = 4·19.

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The effect of ties on the moments of rank criteria

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1. In a recent paper (1956) F. N. David remarked that the moments of any criterion based on ranks could be derived by using Wishart's (1952) tables of sampling formulae for a finite population of N, the K-statistics of such a population being those of the first N natural numbers. In this present note we use this method of approach to find the moments of the mean rank of a sample of n randomly drawn without replacement from a population consisting of the first N natural numbers, ties being allowed. We also obtain the effect of ties on the variance of the sample variance.

2. Where t individuals tie we shall follow Putter (1955) and assign the mean rank in the tie to each individual. We first find the effect of ties on the moments of the finite population. Let the $(p+1)$st to the qth ranked element of the population tie with $(p+1) \leqslant q$. We assign the common rank $m = \frac{1}{2}(p+q+1)$ to each, and the contribution to the rth factorial moment of the population (i.e. the change occasioned by replacing the numbers $p+1, p+2, \ldots, q$ by $m, m, \ldots, m$) is thus

$$\begin{aligned}\Delta\mu_{(r)} &= \frac{1}{N}\left\{tm^{(r)} - \frac{1}{r+1}[(q+1)^{(r+1)} - (p+1)^{(r+1)}]\right\} \\ &= \frac{-1}{N(r+1)}\sum_{j=3}^{r+1}\binom{r+1}{j} m^{(r+1-j)}\left\{\left(\frac{t+1}{2}\right)^{(j)} - \left(\frac{1-t}{2}\right)^{(j)}\right\}. \qquad (1)\end{aligned}$$

Now suppose there are T groups of ties with t_i in the ith group which has common rank $x_i + \frac{1}{2}(N+1)$, for $i = 1, 2, \ldots, T$. The changes in the moments due to each group are additive, and so, after a little reduction, the population central moments are seen to be

$$\left.\begin{aligned}\mu_1' &= \frac{N+1}{2}, \quad \mu_2 = \frac{1}{12N}\left\{N(N^2-1) - \sum_{i=1}^{T} t_i(t_i^2-1)\right\} \\ \mu_3 &= \frac{-1}{4N}\sum_{i=1}^{T} x_i t_i(t_i^2-1), \\ \mu_4 &= \frac{1}{240N}\left\{N(N^2-1)(3N^2-7) - \sum_{i=1}^{T} t_i(t_i^2-1)(3t_i^2-7)\right\} - \frac{1}{2N}\sum_{i=1}^{T} x_i^2 t_i(t_i^2-1).\end{aligned}\right\} \qquad (2)$$

We give the moments rather than the K-statistics since these former are simpler.

3. If it is assumed that two samples n_1 and n_2 are arranged together in ascending order of magnitude, $(N = n_1 + n_2)$, then Wilcoxon's procedure consists of finding the mean of the ranks of one sample, and using this criterion to test for the equivalence of the location parameters of the two populations generating the samples. The moments of the mean of the first sample (ties not admitted), may be written down immediately from Wishart's tables. When ties are allowed, the first moment is unchanged, and the second moment becomes

$$\mu_2(\bar{x}) = \frac{n_1(N-n_1)(N+1)}{n}\left[1 - \frac{1}{N(N^2-1)}\sum_{i=1}^{T} t_i(t_i^2-1)\right]. \tag{3}$$

The third and fourth moments do not simplify at all and they are accordingly most simply computed in any instance by putting numerical values of (2) into Wishart's formulae.

The variance of $\bar{x}$ given in (3) above has previously been derived by Hemelrijk (1952), Kruskal (1952) and Putter (1955). Computations were carried out for varying values of N and n_1, and for different numbers of tied observations. It would appear for such values of n_1 and N for which the normal approximation can be used in significance tests that the effect of ties is not likely to lead to serious error, provided that not more than half the observations are tied.

4. Other procedures have been put forward whereby tests can be carried out for possible differences in the parameters of dispersion of two parent populations. David (1956) suggested the variance of one sample might be used, while Mood (1954) proposed W, the sum of the squares of the deviations of the ranks of one sample from the population mean. It is clear that the same algebraic procedure used above for the sample mean can be used to derive corrections to the moments of both David's and Mood's criterion when ties are allowed. In fact these two criteria are merely the sample variance and the sample sum of squares about $\frac{1}{2}(N+1)$ of a sample from the finite populations whose first four moments are given by (2).

Thus using Abdel-Aty's (1954) general formulae, the first two moments of Mood's statistic become

$$\mathscr{E}(W) = \frac{n_1}{12N}\left\{N(N^2-1) - \sum_{i=1}^{T} t_i(t_i^2-1)\right\},$$

$$\text{var}(w) = \frac{n_1(N-n_1)}{N-1}\{\mu_4 - \mu_2^2\}.$$

For David's statistic, k_2, we have similarly,

$$\mathscr{E}(k_2) = \frac{1}{12(N-1)}\left\{N(N^2-1) - \sum_{i=1}^{T} t_i(t_i^2-1)\right\},$$

$$\text{var}(k_2) = \frac{N(N-n_1)\{(Nn_1 - N - n_1 - 1)(N-1)\mu_4 + (3(N-1)^2 + n_1(N^2-3))\mu_2^2\}}{n_1(n_1-1)(N-1)^2(N-2)(N-3)}.$$

The variances of both these statistics do not simplify if the explicit values of μ_2 and μ_4 from (2) are substituted. It is simplest therefore to compute them, in any given instance, using the numerical values obtained from (2).

5. Although the correction to the variance of the mean has been given by several persons, as far as the writer knows there has been no discussion of corrections for higher moments of the mean, or for other criteria. The general expression in (1) enables as many corrected moments of the population as desired to be calculated, as in (2). Accordingly the only limitations are set by the extent of Wishart's and Abdel-Aty's tables.

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Approximations to the upper 5 % points of Fisher's *B* distribution and non-central χ^2*

BY JOHN W. TUKEY
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1. SUMMARY

Some time ago the writer reported (1949) on an empirical approximation to the upper 5 % point of Fisher's *B* distribution, the distribution of the square root of a non-central χ^2, comparing the approximation with certain of the exact values given by Fisher (1928) and suggesting that the empirical formula might 'be useful for moderate extrapolation'. At about the same time Patnaik (1949), and later Abdel-Aty (1954), studied the problem of approximating the distribution of non-central χ^2. Through the efforts and courtesy of Dr B. I. Harley, of University College, it is now possible to present a comparative table of the results of various approximations, including the streamlined use of moment-fitted Pearson curves by the aid of Table 42 of the new *Biometrika* tables (Pearson & Hartley, 1954).

While the empirical approximation is restricted to the upper 5 % point, only the moment-fitting approach seems likely to give more accurate values in the range considered. Because of its simplicity, the empirical formula may prove useful.

2. DISTRIBUTION AND APPROXIMATIONS

The simplest definition of a non-central χ^2 quantity is

$$x_1^2+x_2^2+\ldots+x_{n-1}^2+(x_n+\beta)^2=\chi'^2=B^2,$$

where the x's are independent unit normals and $\lambda=\beta^2$ is fixed. In discussing the *B* distribution, Fisher used β and $n_1=n$. The empirical approximation to be considered is

$$B_{0.05}\approx 1{\cdot}6449+\beta+0{\cdot}51\frac{n_1-1}{\beta+1}-0{\cdot}024\frac{(n_1-5)(n_1-1)}{\beta(\beta+1)}. \tag{A}$$

(The simplest way to use this for a 5 % point of χ'^2 is to find $B_{0.05}$ and then square the result.)

Patnaik's approximation begins with inverse interpolation in the percentage point tables of χ^2 (first approximation, pp. 207–8) and then follows with a Cornish–Fisher expansion (second approximation, formula (25), page 213). A substantial amount of computation is required.

Abdel-Aty's approximation begins with a cube-root transformation and continues with an expansion of Cornish–Fisher type. The first approximation, that $\{\chi'^2/(n+\lambda)\}^{\frac{1}{3}}=B^{\frac{2}{3}}/(n_1+\beta^2)^{\frac{1}{3}}$ is roughly normally distributed about $1-\sigma^2$ with variance

$$\sigma^2=\frac{2}{9}\frac{n+2\lambda}{(n+\lambda)^2}=\frac{2}{9}\frac{n+2\beta^2}{(n+\beta^2)^2}$$

is quite simple, but the 'closer approximation' involves considerable computation.

The approximation involved in the use of Table 42 of the new *Biometrika* tables corresponds to fitting a Pearson curve to the first four moments of χ'^2, and accepting the percentage points of the fitted Pearson curve.

3. ACCURACY AND COMPARISON

Fisher's table covers $n_1=1\,(1)\,7$ and $\beta=0\,(0{\cdot}2)\,5{\cdot}0$. Comparison of exact and approximate values at a reasonably rude grid is made in Table 1, which covers $n_1=1\,(2)\,7$, $\beta=1\,(2)\,5$. The disagreement is nowhere worse than 0·035 and reaches no more than 0·0037 for $\beta\geqslant 3$. The fit is encouraging.

The comparison between the different methods of approximation is made in Table 2 in terms of non-central χ^2 upper 5 % points, squares of $B_{0.05}$ values. As noted in the summary, this table was undertaken and computed by Dr B. I. Harley.

The values obtained through moments and Table 42 require the computation of the first four moments of χ'^2, and in particular of

$$\beta_1=\frac{8(1+2b)^2}{r(1+b)^3},\quad \beta_2=3+\frac{12(1+3b)}{r(1+b)^2},$$

where

$$r=n+\lambda=n_1+\beta^2 \quad\text{and}\quad b=\lambda/(n+\lambda)=\beta^2/(n_1+\beta^2).$$

Once these values are available, the use of Table 42 is quite simple and direct. It provides approximate values for a moderate selection of percentage points, and its accuracy at the upper 5 % point is excellent.

* Prepared in connexion with research sponsored by the U.S. Office of Naval Research.

Table 1. *Accuracy of approximation* (A)

β	Value	$n_1 = 1$	$n_1 = 3$	$n_1 = 5$	$n_1 = 7$
1	Exact	2·6461	3·1941	3·6291	4·0005
	Approx.	2·6449	3·2029	3·6649	4·0309
3	Exact	4·6449	4·9055	5·1512	5·3840
	Approx.	4·6449	4·9079	5·1549	5·3859
5	Exact	6·6449	6·8162	6·9831	7·1457
	Approx.	6·6449	6·8181	6·9849	7·1453

Table 2. *Approximations to the upper 5 % points of the non-central χ^2 distribution*

$n = n_1$	$\lambda = \beta^2$	Exact	Approx. (A)	Patnaik's approx.		Abdel-Aty's approx.		Moments and Table 42
				1st	2nd	1st	2nd	
2	1	8·64	8·62	8·63	—	8·56	8·38	—
	4	14·64	14·65	14·72	14·67	14·66	14·62	*
	16	33·05+	33·07	33·35	33·06	33·32	33·08	33·07
	25	45·31	45·32	45·66	—	45·64	45·33	—
3	1	10·20	10·26	10·20	—	—	—	—
4	1	11·71	11·87	11·72	—	11·67	11·67	—
	4	17·31	17·36	17·38	17·33	17·34	17·27	*
	16	35·43	35·46	35·69	35·42	35·66	35·44	35·42
	25	47·61	47·64	47·94	—	47·91	47·62	—
7	1	16·00	16·25−	16·01	—	15·98	15·99	—
	4	21·23	21·32	21·28	21·27	21·25	21·21	21·23
	9	28·99	29·01	29·12	—	—	—	—
	16	38·97	38·97	39·16	38·97	39·16	38·96	38·95
	25	51·06	51·06	51·34	—	51·33	51·06	—

* These lie outside the β_1 range for Table 42.

There is some reason to suspect, however, that its accuracy may be close to its best at this particular percentage point.

If other than an upper 5 % point is desired, the method using Table 42 seems simplest and as likely to be accurate as any. For an upper 5 % point, approximation (A) seems simpler and almost as accurate. (The user demanding highest accuracy may well consider applying Table 42 to Abdel-Aty's cube-root transformation, since the formulas for its β_1 and β_2 are given at the foot of his page 538.)

4. Source of approximation (A)

For large β,

$$B^2 = \beta^2\left\{\left(1+\frac{x_n}{\beta}\right)^2 + \frac{x_{n-1}^2+\ldots+x_1^2}{\beta^2}\right\},$$

whence

$$B = \beta\left\{1+\frac{2x_n}{2\beta}+O\left(\frac{1}{\beta^2}\right)\right\} = \beta + x_n + O\left(\frac{1}{\beta}\right),$$

so that the upper 5 % point of $B-\beta$ approaches 1·6449. This fixes the first two terms of (A). The last two came from empirical examination of a table of $B_{0.05}-1{\cdot}6449-\beta$. Presumably similar approximate formulas could be obtained by similar methods from brief tables of other exact percentage points.

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Uniqueness of a result in the theory of accident proneness

By N. L. JOHNSON

(*University College, London*)

1. A simple type of model for accident proneness (see Greenwood & Yule (1920), Arbous & Kerrich (1951), Bates & Neyman (1952), and Arbous & Sichel (1954)) is obtained by supposing

(i) the number of accidents sustained by a given individual in a period of fixed ('unit') length (e.g. 1 year) may be represented by a Poisson variable x with

$$\Pr\{x\} = e^{-\lambda}\frac{\lambda^x}{x!} \quad (x = 0, 1, 2, \ldots),$$

and (ii) λ (>0) varies from individual to individual with a probability density function $p(\lambda)$.

If $p(\lambda)$ be assumed to follow the Pearson type III distribution

$$p(\lambda) = \left(\frac{k}{m}\right)^k \frac{1}{\Gamma(k)} e^{-(k/m)\lambda}\lambda^{k-1} \quad (0 \leqslant \lambda)$$

then the overall distribution of observed number (y) of accidents per individual in a given period of unit length is

$$\begin{aligned}\Pr\{y\} &= \int_0^\infty e^{-\lambda}\frac{\lambda^y}{y!}p(\lambda)\,d\lambda \\ &= \left(\frac{k}{m}\right)^k \frac{1}{\Gamma(k)\,\Gamma(y+1)}\int_0^\infty \lambda^{k+y-1}e^{-(1+k/m)\lambda}\,d\lambda. \\ &= \frac{\Gamma(k+y)}{\Gamma(k)\,\Gamma(y+1)}\left(\frac{k}{m+k}\right)^k\left(\frac{m}{m+k}\right)^y,\end{aligned}$$

i.e. a negative binomial distribution with mean $k(m/k) = m$ and variance $k\frac{m}{k}\left(1+\frac{m}{k}\right) = m+\frac{m^2}{k}$.

2. Now, let (y, z) be the numbers of accidents sustained by the *same* individual in two successive periods of unit length. Assuming that for each individual λ retains the same value in the second period as it possesses in the first, we find, using Bayes' theorem,

$$p(\lambda \mid y) = \frac{e^{-\lambda}\lambda^y p(\lambda)}{\int_0^\infty e^{-\lambda}\lambda^y p(\lambda)\,d\lambda}. \tag{1}$$

Hence

$$\begin{aligned}\Pr\{z \mid y\} &= \int_0^\infty e^{-\lambda}\frac{\lambda^z}{z!}p(\lambda \mid y)\,d\lambda \\ &= \frac{1}{\Gamma(z+1)}\frac{\int_0^\infty \lambda^{k+y+z-1}e^{-(2+k/m)\lambda}\,d\lambda}{\int_0^\infty \lambda^{k+y-1}e^{-(1+k/m)\lambda}\,d\lambda} \\ &= \frac{\Gamma(k+y+z)}{\Gamma(k+y)\,\Gamma(z+1)}\left(\frac{m+k}{2m+1}\right)^{k+y}\left(\frac{m}{2m+k}\right)^z.\end{aligned}$$

The conditional distribution of z, given y, is therefore a negative binomial with mean $(k+y)\frac{m}{m+k}$ and variance

$$(k+y)\frac{m}{m+k}\left(1+\frac{m}{m+k}\right)=\frac{m(k+y)(2m+k)}{(m+k)^2}.$$

The regression of z on y is linear (see also Arbous & Sichel (1954)).

It may also be noted that $\mathscr{E}(z \mid y=0)=mk/(m+k)$, while $\mathscr{E}(z)=\mathscr{E}(y)=m$, so that selection of individuals free from accidents in the first period reduces the accident rate to be expected in the second period by a factor $k/(m+k)$.

3. It is the purpose of this note to show that if $p(\lambda)$ is not a type III probability density function, the regression of z on y is not linear.

Since $\mathscr{E}(z \mid \lambda)=\lambda$ it follows that

$$\mathscr{E}(z \mid y)=\int_0^\infty \lambda p(\lambda \mid y)\,d\lambda.$$

Hence, from (1)

$$\mathscr{E}(z \mid y)=\int_0^\infty e^{-\lambda}\lambda^{y+1}p(\lambda)\,d\lambda \Big/ \int_0^\infty e^{-\lambda}\lambda^{y}p(\lambda)\,d\lambda$$

and so, for the regression of z on y to be linear, we must have

$$\alpha+\beta y=\frac{\int_0^\infty \lambda^{y+1}e^{-\lambda}p(\lambda)\,d\lambda}{\int_0^\infty \lambda^{y}e^{-\lambda}p(\lambda)\,d\lambda}, \tag{2}$$

where α and β are constants satisfying $\alpha+\beta m=m$ (since $\mathscr{E}(z)=\mathscr{E}(y)=m$) and $\beta>0$ (since $\mathscr{E}(z \mid y)\geqslant 0$ for all y).

Equation (2) can be rewritten

$$\alpha+\beta y=\frac{\mu'_{y+1}}{\mu'_y}, \tag{3}$$

where μ'_r is the rth moment about zero of the probability density function

$$e^{-\lambda}p(\lambda)\Big/\int_0^\infty e^{-\lambda}p(\lambda)\,d\lambda \quad (\lambda>0). \tag{4}$$

Since $\mu'_0=1$, it follows that

$$\mu'_1=\alpha>0, \quad \mu'_2=\alpha(\alpha+\beta), \quad \mu'_3=\alpha(\alpha+\beta)(\alpha+2\beta),$$

and in general

$$\mu'_r=\alpha(\alpha+\beta)\ldots(\alpha+(r-1)\beta).$$

Hence $\mu'_r<(\beta'_r)^r$ and so $\overline{\lim_{r\to\infty}}\,(\mu'^{1/r}_r/r)<\beta'$, where $\beta'=\alpha+\beta$.

Hence the moments $\{\mu'_r\}$ determine the distribution (4) uniquely; and $p(\lambda)$ is, therefore, also determined uniquely by (2). A function $p(\lambda)$ of type III form does satisfy (2), and this is therefore the *only* form of probability density function satisfying (2).

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A note on the mean deviation of the binomial distribution

By N. L. JOHNSON
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In a paper dealing with the mathematical theory of risk in insurance Gruder (1930) has shown that

$$\begin{aligned}\sum_{r=m}^{n}\binom{n}{r}p^rq^{n-r}(r-np) &= \sum_{r=m}^{n}\binom{n}{r}p^rq^{n-r}[rq-(n-r)p]\\ &= \sum_{r=m}^{n}\left[n\binom{n-1}{r-1}p^rq^{n-r+1}-n\binom{n-1}{r}p^{r+1}q^{n-r}\right]\\ &= n\binom{n-1}{m-1}p^mq^{n-m+1}\\ &= m\binom{n}{m}p^mq^{n-m+1} \qquad (1)\end{aligned}$$

This result may be applied to the evaluation of the mean deviation of a binomial distribution, an application which, it is believed, is new. The mean deviation of the binomial distribution is equal to

$$\sum_{r=0}^{n}\binom{n}{r}p^rq^{n-r}\,|\,r-np\,| = -\sum_{r=0}^{np+\zeta-1}\binom{n}{r}p^rq^{n-r}(r-np)+\sum_{r=np+\zeta}^{n}\binom{n}{r}p^rq^{n-r}(r-np),$$

where $np+\zeta$ is the least integer greater than np.

Since

$$\sum_{r=0}^{n}\binom{n}{r}p^rq^{n-r}(r-np)=0,$$

$$\sum_{r=0}^{n}\binom{n}{r}p^rq^{n-r}\,|\,r-np\,| = 2\sum_{r=np+\zeta}^{n}\binom{n}{r}p^rq^{n-r}(r-np).$$

Hence, using (1), the mean deviation of the binomial distribution is

$$2(np+\zeta)\binom{n}{np+\zeta}p^{np+\zeta}q^{nq-\zeta+1}. \qquad (2)$$

The ratio of mean deviation to standard deviation is

$$R=\frac{2(np+\zeta)}{\sqrt{n}}\binom{n}{np+\zeta}p^{np+\zeta-\frac{1}{2}}q^{nq-\zeta+\frac{1}{2}}. \qquad (3)$$

Applying Stirling's formula in the form

$$N!\doteqdot\sqrt{(2\pi)}\,N^{N+\frac{1}{2}}e^{-N}\left(1+\frac{1}{12N}\right)$$

we obtain

$$R\doteqdot\sqrt{\frac{2}{\pi}}\left(1+\frac{\zeta}{np}\right)^{\frac{1}{2}-\zeta}\left(1-\frac{\zeta}{nq}\right)^{\zeta-\frac{1}{2}}\left(1+\frac{\zeta}{np}\right)^{-np}\left(1-\frac{\zeta}{nq}\right)^{-nq}\left(1+\frac{1}{12n}\right)\left(1+\frac{1}{12(np+\zeta)}\right)^{-1}\left(1+\frac{1}{12(nq-\zeta)}\right)^{-1}.$$

If np and nq are large

$$\left(1+\frac{\zeta}{np}\right)^{-np}\doteqdot e^{-\zeta}\left(1+\frac{\zeta^2}{2np}\right),$$

$$\left(1-\frac{\zeta}{nq}\right)^{-nq}\doteqdot e^{\zeta}\left(1+\frac{\zeta^2}{2nq}\right),$$

$$\left(1+\frac{\zeta}{np}\right)^{\frac{1}{2}-\zeta}\doteqdot 1+\frac{\zeta(\frac{1}{2}-\zeta)}{np},$$

$$\left(1-\frac{\zeta}{nq}\right)^{\zeta-\frac{1}{2}}\doteqdot 1+\frac{\zeta(\frac{1}{2}-\zeta)}{nq},$$

and

$$\begin{aligned}R&\doteqdot\sqrt{\frac{2}{\pi}}\left(1+\frac{\zeta^2}{2npq}+\frac{\zeta(\frac{1}{2}-\zeta)}{npq}+\frac{1}{12n}-\frac{n}{12(np+\zeta)(nq-\zeta)}\right)\\ &\doteqdot\sqrt{\frac{2}{\pi}}\left(1+\frac{\zeta(1-\zeta)}{2npq}-\frac{1-pq}{12npq}\right). \qquad (4)\end{aligned}$$

The table below compares the exact formula (3) with the approximate formula (4) for some typical values of n and p. The ratio takes the same value for $p = x$ and $p = 1-x$, so values of p are restricted to the range $0{\cdot}1 - (0{\cdot}1) - 0{\cdot}5$. It will be noted that the Normal limiting value of $\sqrt{(2/\pi)} = 0{\cdot}7979$ is approached quite rapidly as n increases. The approximate formula (4) gives nearly three figure accuracy for $0{\cdot}2 \leqslant p \leqslant 0{\cdot}8$, while for $n \geqslant 50$ the exact and approximate values agree to four decimal places.

n \ p	0·1		0·2		0·3		0·4		0·5	
	(3)	(4)	(3)	(4)	(3)	(4)	(3)	(4)	(3)	(4)
10	0·7351	0·7313	0·7640	0·7630	0·7733	0·7729	0·7771	0·7768	0·7782	0·7779
20	0·7652	0·7642	0·7807	0·7804	0·7855	0·7854	0·7874	0·7874	0·7879	0·7879
50	0·7846		0·7909		0·7929		0·7937		0·7939	
100	0·7912		0·7949		0·7954		0·7958		0·7959	

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A comment on D. V. Lindley's statistical paradox

By M. S. BARTLETT
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I read with considerable interest the discussion by D. V. Lindley (1957), demonstrating the possibility of contradiction between the result of a statistical significance test and an assessment of the posterior probability of a null hypothesis. I would agree that he establishes the point that one must be cautious when using a fixed significance level for testing a null hypothesis irrespective of the size of sample one is taking. However, there is a slip, in his expression for K under his equation (1), that appears to me, unless corrected, to lead to an overstatement of this point. The prior distribution for θ, given that $\theta \neq \theta_0$, was assumed to be uniform over an interval I, and hence its density function should be $1/I$ in this interval. This leads to the extra factor $1/I$ in the second term in the expression for K.* This expression then becomes consistent with Jeffreys's equation (10), § 5·0, in his book (second edition, 1948).

The occurrence of I in the formula for the posterior probability $\bar{c}$ of the null hypothesis θ_0, this quantity $\bar{c}$ satisfying approximately the relation

$$\frac{\bar{c}}{1-\bar{c}} = \frac{c}{1-c}\left[\frac{I}{\sigma}\sqrt{\left(\frac{n}{2\pi}\right)}\exp\left\{\frac{-n(\bar{x}-\theta_0)^2}{2\sigma^2}\right\}\right], \tag{1}$$

where c is the prior probability of θ_0, now makes the value of $\bar{c}$ much more arbitrary. In fact, in situations where one might be tempted to put I infinity the silly answer $\bar{c} = 1$ ensues. D. V. Lindley has suggested to me, in correspondence, that one way out of this dilemma would be to make $c/(1-c)$ the prior odds in favour of the null hypothesis against *any unit interval* of the alternative values, but this is rather an artificial evasion of the difficulty. It is common for those who use the Bayes's approach to assume a uniform prior distribution for the mean of a normal population. If the difference in means between two populations (for simplicity, of known equal variances) is considered, the question might legitimately be asked whether these populations, from each of which a sample is available, are identical. The most natural prior probabilities would seem to me, if we try to use the Bayes's approach, to be c for this null hypothesis, and a remaining uniform prior distribution of the true difference in population means over the entire infinite range.

The other point that might be noticed about formula (1), if we disregard the above difficulty and agree to leave I finite, is this. Certainly, for a fixed significance level (and I and σ fixed), the posterior odds increase with $\sqrt{n}$. But from the Neyman-Pearson theory of the power of tests, if a null hypothesis θ_0 is being tested against a single alternative θ_1, the sample size n would if possible be chosen in relation to the 'distance' $d \equiv \theta_1 - \theta_0$ between the two hypotheses ($\sqrt{n}$ inversely proportional to d). The situation under discussion is a little more complicated but, with a range of I for the alternatives, it would be fairly

reasonable to choose the sample size n analogously, making $\sqrt{n}$ proportional to $1/I$. If we write $\sqrt{n} = A\sigma/I$, we obtain

$$\frac{\bar{c}}{1-\bar{c}} = \frac{c}{1-c}\left\{\frac{A}{\sqrt{(2\pi)}}e^{-\frac{1}{2}\lambda^2}\right\}, \tag{2}$$

where $\lambda = \sqrt{n}(\bar{x}-\theta_0)/\sigma$; in (2) there is a constant relation between $\bar{c}$ and λ for fixed c.

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LINDLEY, D. V. (1957). A statistical paradox. *Biometrika*, **44**, 187–92.

Editorial Note. Mr Lindley agrees and apologizes for the fact that a factor $1/I$ was omitted from his equation (1), but points out that in the two examples which he discusses this factor is unity. His general argument as to the limiting value of $\bar{c}$ is in any case unaffected, and his two particular examples are also unaffected. There appears to be no real difference of opinion between Prof. Bartlett and Mr Lindley on this point.

The point raised by Prof. Bartlett's second paragraph is related to the difficulty of laying down a uniform prior probability for a parameter of infinite range, a point which, in my opinion, has not been properly cleared up; if the probability of μ is $k\,d\mu$, integration over the infinite range leads to the conclusion that k is zero (the integral having to be unity). The root of this difficulty seems to be that several limiting processes are involved and no clear rules have been laid down as to which, if any, has priority. In any case, this point mainly concerns estimation, whereas Mr Lindley was concerned with testing hypotheses.

In regard to Prof. Bartlett's final point, it may be useful to observe that some procedure of the type he suggests is implicit in the idea of the asymptotic relative efficiency of a test. In general, the power of a test against a specific alternative tends to unity with increasing sample size. To compare two tests asymptotically at a fixed significance level, it is necessary to allow the alternative to approach the null hypothesis as the sample size increases. M.G.K.

* There is also a further dropping of a factor $1/\sigma$ in the last formula on p. 191, but this is a more trivial slip.

CORRIGENDA

Biometrika (1957), **44**, pp. 150–8

'The use of a concomitant variable in collecting an experimental design.' By D. R. Cox

Dr K. R. Nair has kindly pointed out that some of the results for Methods II and V of the above paper have been given by him in *Sankhya* (1942), **6**, 167–174. He has also noted that in formula (6) of my paper $(\bar{x}_i-\bar{x}_j)^2$ should read $k(\bar{x}_i-\bar{x}_j)^2$. D.R.C.

Biometrika (1957), **44**, pp. 168–78

'Multiple runs.' By D. E. BARTON and F. N. DAVID

P. 174, line 10. *Delete* 'transition probabilities' and *substitute* 'joint probabilities of two successive events'.

P. 174, line 13. *Delete* the full stop after the expression for W and *add* 'and where $p_i = 1/k$, $(i = 1, 2, \ldots, k)$'.

REVIEWS

Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability. Edited by JERZY NEYMAN. University of California Press, for whom Cambridge University Press act as agents. 1957. Vol. I. **Theory of Statistics,** pp. 208, 45*s*. Vol. II: **Probability Theory,** pp. 246, 49*s*. Vol. III: **Astronomy and Physics,** pp. 252, 47*s*. Vol. IV: **Biology and Problems of Health,** pp. 179, 43*s*. 6*d*. Vol. V: **Econometrics, Industrial Research and Psychometry,** pp. 184, 43*s*. 6*d*.

Journal space and the limited versatility of the reviewer would preclude anything like a critical appraisal of these books in which are printed all the papers presented at the third symposium held in 1954/55 at the Statistical Laboratory of the University of California at Berkeley. I propose therefore to write a 'contents' review in order to show that there is something for everyone in the reported proceedings.

Volume I. There are papers by: J. Berkson, 'Estimation by Least Squares and by Maximum Likelihood'; Z. W. Birnbaum, 'On a Use of the Mann-Whitney Statistic'; H. Chernoff and H. Rubin, 'The Estimation of the Location of a Discontinuity in Density'; A. Dvoretzky, 'On Stochastic Approximation'; S. Ehrenfeld, 'Complete Class Theorems in Experimental Design'; G. Elfving, 'Selection of Nonrepeatable Observations for Estimation'; U. Grenader and M. Rosenblatt, 'Some Problems in Estimating the Spectrum of a Time Series'; J. L. Hodges & E. L. Lehmann, 'Two Approximations to the Robbins-Monro Process'; W. Hoeffding, 'The Role of Assumptions in Statistical Decisions'; S. Karlin, 'Decision Theory for Pólya Type Distributions. Case of Two Actions': L. le Cam, 'On the Asymptotic Theory of Estimation and Testing Hypotheses'; H. Robbins, 'An Empirical Bayes Approach to Statistics'; M. Rosenblatt, 'Some Regression Problems in Time Series Analysis'; C. Stein, 'Efficient Nonparametric Testing and Estimation' and 'Inadmissibility of the Usual Estimator for the Mean of a Multivariate Normal Distribution'; B. L. van der Waerden, 'The Computation of the X-distribution'.

The majority of the papers are on technical points and in the mathematical style familiar to readers of the *Annals of Mathematical Statistics*. The most interesting paper to the reviewer is that by Dr Berkson who with his commonsense approach and tenacity of purposes refuses to be blinded by mathematics and who in this paper raises once again the challenge in the problem.

Volume II. There are papers by: D. Blackwell, 'On a Class of Probability Spaces'; S. Bochner, 'Stationarity, Boundedness, Almost Periodicity of Random-Valued Functions'; K. L. Chung, 'Foundations of the Theory of Continuous Parameter Markov Chains'; A. H. Copeland, 'Probabilities, Observations and Predictions'; J. L. Doob, 'Probability Methods Applied to the First Boundary Value Problem'; R. M. Fortet, 'Random Distributions with an Application to Telephone Engineering'; J. M. Hammersley, 'The Zeros of a Random Polynomial'; T. E. Harris, 'The Existence of Certain Markov Processes'; K. Itô, 'Isotropic Random Current'; P. Lévy, 'A Special Problem of Brownian Motion, and a General Theory of Gaussian Random Functions'; M. Loève, 'Ranking Limit Problem'; E. Lukacs, 'Characterization of Populations by Properties of Suitable Statistics'; K. Menger, 'Random Variables from the Point of View of a General Theory of Variables'; E. Mourier, 'Random Elements in Banach Spaces'; R. Salem and A. Zygmund, 'Random Trigonometric Polynomials'.

The papers in this volume appear to be written by probabilists for the benefit of other probabilists and most of them reflect the current interest of probabilists in the random process. There is no immediate statistical link-up—there is no reason why there should be—and applied statisticians generally will find this volume unrewarding. The theoretical statistician seeking to apply the random process may find a study of it useful.

Volume III. This volume is divided into two sections, one for astronomy and one for physics. The contributions to astronomy consists of (i) Hertzsprung-Russell Diagram, with papers by: O. J. Eggen, 'The Relationship between the Color and the Luminosity of Stars near the Sun'; J. L. Greenstein, 'The Spectra and other Properties of Stars Lying Below the Normal Main Sequence'; H. L. Johnson, 'Photoelectric Studies of Stellar Magnitudes and Colors'; G. E. Kron, 'Evidence for Sequences in the Color-Luminosity for M-Dwarfs'; B. Strömgren, 'The Hertzsprung-Russell Diagram'; and of (ii) Spatial Distribution of Galaxies, with papers by: G. C. McVittie, 'Galaxies, Statistics and Relativity'; J. Neyman, E. L. Scott and C. D. Shane, 'Statistics of Images of Galaxies'; F. Zwicky, 'Statistics of Clusters of Galaxies'. The contributions to physics are from: A. Blanc-Lapierre and A. Tortrat,

'Statistical Mechanics and Probability Theory'; M. Kac, 'Foundations of Kinetic Theory'; E. Montroll, 'Theory of the Vibration of Simple Cubic Lattices'; and N. Weiner, 'Nonlinear Prediction and Dynamics'.

Statisticians of all types will find this volume of interest and need not be deterred by a lack of knowledge of astronomy. The papers are well and clearly written, and it appears possible from them to get some idea of work being done by statisticians in fields other than those we think of as orthodox (i.e. biology, economics, etc.)

Volume IV. There are papers by: J. Crow and M. Kimura, 'Some Genetic Problems in Natural Populations'; E. R. Dempster, 'Some Genetic Problems in Controlled Populations'; J. Neyman, T. Park and E. L. Scott, 'Struggle for Existence. The *Tribolium* Model'; M. S. Bartlett, 'Deterministic and Stochastic Models for Recurrent Epidemics'; A. T. Bharucha-Reid, 'On the Stochastic Theory of Epidemics'; C. L. Chiang, J. L. Hodges and J. Yerushalmy, 'Statistical Studies in Medical Diagnoses'; J. Cornfield, 'A Statistical Problem Arising from Retrospective Studies'; D. G. Kendall, 'Deterministic and Stochastic Epidemics in Closed Populations'; W. F. Taylor, 'Problems in Contagion'.

The level of these papers in this volume is high. They are written in a mathematical language that all statisticians should be able to understand, and since there is a concrete problem on which the mathematics is hung there is a sense of purpose about them which is often lacking in papers such as those, for example, in Volume I. The fields of application are the traditional biometric ones and in consequence this volume will be needed by many workers in the statistical field.

Volume V. Under the subtitle 'Contributions to Econometrics' there are papers by: K. J. Arrow and L. Hurwicz, 'Reduction of Constrained Maxima to Saddle-Point Problems'; E. W. Barankin, 'An Objectivistic Theory of Probability'; C. W. Churchman, 'Problems of Value Measurement for a Theory of Induction and Decisions'; P. Suppes, 'The Role of Subjective Probability and Utility in Decision-Making'. Under the subtitle 'Contributions to Industrial Research' there are papers by: A. H. Bowker, Continuous Sampling Plans'; C. Daniel, 'Fractional Replication in Industrial Research'; M. Sobel, 'Sequential Procedures for Selecting the Best Exponential Population'. Under the subtitle 'Contributions to Psychometry' there are papers by: T. W. Anderson and H. Rubin, 'Statistical Inference in Factor Analysis'; F. Mosteller, 'Stochastic Learning Models'; and H. Solomon, 'Item Analysis and Classification Techniques'.

F. N. DAVID

Fejezetek a klasszikus valószínűségszámításból. (Chapters on the classical probability calculus.) By JORDAN KÁROLY (Charles Jordan). Budapest: Akadémiai Kiado. 1956. Pp. 616. 120 forint.

This book presents the theory of probability as the famous Hungarian statistician sees it after 50 years work and 30 years lecturing. Big changes have taken place during that time; but as the new point of view is set out in a book by Rényi, the author here concentrates on the classical methods and especially their practical applications. The book begins with introductory and historical remarks, definitions and a long chapter on mathematical techniques. It then goes on to arithmetic and geometric expectations, the binomial, Poisson and Lexis distributions in one and two variables, applications to games, gambler's ruin, geometric probabilities, the theory of errors and least squares, and the kinetic theory of gases. The book is beautifully printed and well indexed, and includes a short table of binomial coefficients. Hungarians are very lucky to have such a volume in their language, although unfortunately this means that other nationalities will be effectively deprived of the use of the book.

CEDRIC A. B. AND PIRI SMITH

Mathematische Statistik (Band LXXXVII of the Grundlehren der Mathematische Wissenschaften series). By B. L. VAN DER WAERDEN. Berlin: Springer Verlag. 1957. Pp. 360. D.M. 49.60 (£4. 9*s*.).

It is interesting to compare this book with another of the same title by Schmetterer, written earlier in 1956, and published by Springer in Vienna (see *Biometrika*, **44**, 293). Van der Waerden achieves a wider range of topics in a shorter space by adopting a more informal and less systematic approach than Schmetterer and by going less deeply into the general theory. Thus, although he is careful to preserve mathematical rigour, the author treats the commoner, simpler and more frequently applied

tests and estimations problems. He opens with a simple presentation of Kolmogoroff's axioms and follows with an examination of frequency distributions and the information contained in their means and variances. He then treats Kolmogoroff's K and allied measures and develops the mathematical theory of transformation of variates and characteristic functions. Next comes the theory of distributions derived from the normal followed by two short chapters on estimation by least squares and maximum likelihood and minimum χ^2. A chapter on bioassay intervenes before consideration of simpler tests and the bare outlines of the Neyman–Pearson theory. The last two chapters are concerned with Wilcoxon's and allied tests and with correlation, both product moment and by rank methods.

The book is beautifully produced, as indeed it should be at its formidable price. D. E. BARTON

Statistical Methods in Research and Production with special reference to the Chemical Industry (third edition revised). Edited by O. L. DAVIES. Edinburgh: Oliver and Boyd Ltd. 1957. Pp. 396. 45*s*.

This book is the third edition of one first published in 1947, and runs to 390 pages against 286, while of eight authors listed five names are common to the seven listed in the first edition. (The reviewer has not seen the second edition). The potential reader should not be put off by the expression 'with special reference to the chemical industry' if that does not cover his own field, as all this means is that the examples are mostly chosen from chemical subjects.

The book consists mainly of a practical exposition of the logic and application of the more elementary statistical methods. Personally I would consider that the treatment is weighted somewhat too much on the application side, since in my experience amateur statisticians rarely fail to apply methods correctly, but are often weak in the logic of what they are doing. In any statistical method there is quite a narrow range of readers between those who know it already and those who are unwilling to make the effort to learn it, and everyone tends to have different ideas on what the typical reader should be assumed to know, but I feel that actual readers will mostly be less logical and more arithmetical than is apparently assumed by the authors.

The methods of exposition and the notation are admirably clear. Trying out on myself a section describing a method which I have not used I had no difficulty at all in understanding and applying it.

The only part of the book I disagree with to any serious extent is the statement on page 1 that 'statistics may be defined as the study of chance variations'. It is surely truer to say that statistics is the study of how to draw valid conclusions from numerical data in spite of interference from chance variations.

The scatter diagram on page 191, said to represent a low correlation, in fact corresponds to one of approximately 0·584.

The book may be strongly recommended to anyone, even with very little mathematical knowledge, wishing to apply statistical methods to almost any form of industrial research or production.

L. MCMULLEN

Statistical Methods in Quality Control. By D. J. COWDEN. New York: Prentice-Hall Inc. 1957. Pp. xxiv + 727. $12.00.

This book is extremely comprehensive although as a result it is unfortunately very long. It covers not only all the common methods of quality control, but also gives the basic theory of statistical methods required going somewhat further in this direction than would be needed in order to carry out just the basic techniques in quality control.

The first eleven chapters deal with statistical theory starting with measures of location, dispersion and shape, following with the elements of probability theory, exemplified by the binomial theorem, and leading up to the normal curve. This is followed by methods of estimation, tests of hypotheses and confidence limits together with simple analyses of variance and tests for the homogeneity of variances. The remaining twenty-nine chapters deal with quality control, interspersed with some chapters on portions of statistical theory, such as frequency curves, the hypergeometric distribution and regression techniques. Single, double and multiple inspection schemes are covered and the concepts of the operating characteristic and average outgoing quality are dealt with fully. There is a very interesting chapter on the economics of control charts, although it is a pity that the discussion is restricted to the case where the inspection scheme is based on single samples. The book does not deal in any detail with continuous inspection schemes although Dodge's scheme where sampling is reduced until a defective

appears is mentioned. The schemes based on the combination of small samples, such as the deferred sentencing schemes, or the schemes using both warning and outer limits in combination, suggested by B. P. Dudding and W. J. Jennett, and further investigated by E. S. Page, are not discussed. The exposition throughout is very clear and all the explanations given are extremely comprehensive. It is refreshing to find that the author faces up to such problems as non-normality and deals with them realistically.

There are twenty-seven appendices containing useful tables for many of the tests and techniques discussed in the book including such distributions as the studentized range, W. J. Dixon's ratios for testing extreme values, the circular autocorrelation coefficient with lag 1 and certain percentage points of the Gram–Charlier curve.

This book would probably make a difficult course in statistics for a student to work through on his own because it is very hard to see the wood from the trees. The student would also be left in some difficulty as to which of the many techniques available he should apply in any given practical problem. The volume would, however, be very useful as a textbook for a course on quality control when the teacher could sign-post the way and use much of the material in the book for illustration and consolidation. As a refresher and reference work, too, the book should have an assured place.

P. G. MOORE

Nonparametric Statistics for the Behavioral Sciences. By S. SIEGEL. London: McGraw-Hill Publishing Co. Ltd. 1956. Pp. 313. 49*s*.

It is stated that this book will be of 'special interest to practising behavioral scientists, practising statisticians, and to somewhat advanced students of these fields'. Actually as one of the McGraw-Hill Series in Psychology it is undoubtedly aimed at psychologists and very useful it should prove to them. What Mr Siegel has done is to collect together a number of (so-called) non-parametric tests which are either in common use or are obviously of use in the analysis of psychological data and to divide them up into approximately appropriate compartments. Thus, we are given in the one sample case, the binomial test, χ^2, Kolmogoroff–Smirnoff, and a runs test. For two samples we have Wilcoxon, Wald, Randomization and the various tests for a 2×2 table. The several procedures are quite well set out and are illustrated by numerical examples. There is no real attempt at mathematical derivations.

Neyman and Pearson in introducing their extremely useful concept of a power function, made possible the development of a number of new ideas in what, for the sake of distinction, I might call parametric statistics. But while in the mathematical development it is desirable to represent the hypothesis alternate to that tested by some sort of functional form, in practice one is usually without any clear idea of what the alternative hypothesis may be, and one is lucky if it is possible to assign it to a broad class. Non-parametric statistical tests are merely tests for randomness and the ways in which data can depart from randomness are infinite. It is only possible therefore to specify very broadly the form in which deviations from randomness can take in order to help with the choice of a critical region for the test statistic and to decide which test statistic would be the more appropriate. A few general principles can, however, be adopted if one uses commonsense. For example, a criterion based on ranks will usually be more sensitive than a criterion based on the grouping of observations (such as runs), provided like is compared with like.

Mr Siegel does not really come to grips with this business of the comparison of tests. He uses a concept called *power-efficiency* which lends a spurious depth to some of his discussions without really contributing a great deal to one's ideas about the situations suitable for the application of the tests which he proposes. This is a pity because, as the reviewer has tried to indicate, the application of non-parametric tests must of necessity be based on commonsense rather than on the mathematical development of the alternate hypothesis. His book cannot therefore be recommended as other than a useful summary of some parametric tests in everyday use.

F. N. DAVID

Scientific Inference (second edition). By SIR HAROLD JEFFREYS. London: Cambridge University Press. 1957. Pp. 236. 25*s*.

The first edition of this book was published in 1931, re-issued with addenda in 1937, and now is presented largely re-written and developed. It was reviewed in this journal in 1932. To those familiar with Sir Harold's book on probability and with the first edition of this book, the present edition holds no

surprises; the author takes the point of view which we have come to expect. But whether we agree or not with the ideas put forward on probability (direct and inverse), or sampling, or significance, to mention a few among many topics, the writing is stimulating and a challenge to the reader. It is books such as these which are the leaven in the dough of academic textbooks.

Introduction to Statistical Analysis (second edition). By WILFRED J. DIXON and J. MASSEY Jr. London: McGraw-Hill Publishing Co. Ltd. 1957. Pp. 481. 45*s*.

This is the second edition of a textbook which was reviewed by John Wishart in 1952 (*Biometrika*, Vol. 39). There are some changes and additions. The chapters on 'Various Measures of Central Value and Dispersion', on 'Statistical Inference' and on 'Analysis of Variance' have been re-written and enlarged. A small amount of material has been interpolated in other chapters, and a completely new chapter on probability has been added at the very end. The quantity of statistical tables given in the first edition was impressive and these have now been increased both in number and extent. The present writer would agree with Wishart that 'the book under review compares favourably with others of its kind'. It is, however, somewhat stereotyped and uninspiring.

OTHER BOOKS RECEIVED

1. **Proceedings of International Symposium on Algebraic Number Theory,** held Tokyo and Nikko, 1955. Science Council of Japan. 1956. Pp. xx + 267.

2. **Dictionary of Statistical Terms.** By M. G. KENDALL and W. R. BUCKLAND. Edinburgh: Oliver and Boyd, for the International Statistical Institute. 1957. Pp. ix + 493. 25*s*.

3. **Psychological Tests and Personnel Decisions.** By L. J. CRONBACH and G. C. GLESER. University of Illinois Press. 1957. Pp. 165. $3.50.

4. **Nonparametric Methods in Statistics.** By D. A. S. FRASER. New York: John Wiley and Sons Inc.; London: Chapman and Hall. 1957. Pp. 299. 68*s*.

5. **Introduction to Operations Research.** By C. W. CHURCHMAN, R. L. ACKOFF and E. L. ARNOFF. New York: John Wiley and Sons Inc.; London: Chapman and Hall. 1957. Pp. 645. $12.00.

6. **Statistical Analysis of Stationary Time Series.** By V. GRENANDER and M. ROSENBLATT. New York: John Wiley and Sons Inc.; London: Chapman and Hall. 1957. Pp. 300. 88*s*.

7. **Models of Man.** By H. A. SIMON. New York: John Wiley and Sons Inc.; London: Chapman and Hall. 1957. Pp. 287. 40*s*.

8. **Individual Differences in Night-Vision Efficiency.** Medical Research Council Special Report No. 294. London: Her Majesty's Stationery Office. 1957. Pp. 83. 8*s*.

9. **Sources and Nature of the Statistics of the United Kingdom,** Vol. 2. By M. G. KENDALL. Edinburgh: Oliver and Boyd, for the Royal Statistical Society. 1957. Pp. 343. 30*s*.

10. **Regression Analysis.** By P. LYLE (third edition revised by L. H. C. TIPPETT). Edinburgh: Oliver and Boyd. 1957. Pp. xiii + 204. 16*s*.

11. **Teach yourself Statistics.** By R. GOODMAN. London: English Universities Press Ltd. 1957. Pp. 238. 7*s*. 6*d*.

12. **Life and Other Contingencies,** Vol. 2. P. F. HOOKER and L. H. LONGLEY-COOK. Cambridge: Published for the Institute of Actuaries at the University Press. 1957. Pp. 256. 20*s*.

13. **Tables of the Normal Probability Integral.** By D. B. OWEN. Washington, D.C.: Department of Commerce. 1957. Pp. 58. 40 cents.

14. **The Bivariate Normal Probability Distribution.** By D. B. OWEN. Washington, D.C.: Department of Commerce. 1957. Pp. 136. 65 cents.

15. **Statistics: a New Approach.** By W .ALLEN WALLIS and H. V. ROBERTS. London: Methuen and Co. Ltd. 1957. Pp. 646. 50*s*. [American Edition reviewed *Biometrika*, **44**, p. 297.]

16. **Voluntary Health Insurance in Two Cities.** By O. W. ANDERSON. Harvard University Press; London: Oxford University Press. 1957. Pp. 145. 40*s*.

17. **American Housing and its Uses.** By L. WINNICK. New York: John Wiley and Sons Inc.; London: Chapman and Hall. 1957. Pp. 143. 44*s*.

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